1 Introduction

In the following let $K = \mathbb{R}$ or $K = \mathbb{C}$.

**Definition 1.1** Let $X_0, X_1$ be Banach spaces over $K$. Then the pair $(X_0, X_1)$ is called **admissible, compatible or an interpolation couple** if there is a Hausdorff topological vector space $Z$ such that $X_0, X_1 \hookrightarrow Z$ with continuous embeddings.

**Lemma 1.2** Let $(X_0, X_1)$ be an admissible pair of Banach spaces. Then $X_0 \cap X_1$ normed by

$$
\|x\|_{X_0 \cap X_1} = \max(\|x\|_{X_0}, \|x\|_{X_1}), \quad x \in X_0 \cap X_1,
$$

and $X_0 + X_1$ normed by

$$
\|x\|_{X_0 + X_1} = \inf_{x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1} \|x_0\|_{X_0} + \|x_1\|_{X_1}, \quad x \in X_0 + X_1,
$$

are Banach spaces.

**Notation:** In the following we write for simplicity $T \in \mathcal{L}(X_j, Y_j)$, $j = 0, 1$, for the condition that $T: X_0 + X_1 \to Y_0 + Y_1$ is a linear operator with $T|_{X_j} \in \mathcal{L}(X_j, Y_j)$ for $j = 0$ and $j = 1$.

**Definition 1.3** Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be admissible pairs of Banach spaces and let $X, Y$ be Banach spaces.

1. $X$ is called **intermediate space** with respect to $(X_0, X_1)$ if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$ with continuous embeddings.
2. **X** and **Y** are called interpolation spaces with respect to \((X_0, X_1)\) and \((Y_0, Y_1)\) if **X** and **Y** are intermediate spaces with respect to \((X_0, X_1)\) and \((Y_0, Y_1)\), respectively, and if

\[ T \in \mathcal{L}(X_j, Y_j), j = 0, 1, \implies T|_X \in \mathcal{L}(X, Y) \]

for all \(T: X_0 + X_1 \rightarrow Y_0 + Y_1\) linear.

3. **X** is called interpolation space with respect to \((X_0, X_1)\) if the previous conditions hold with \(X = Y, (X_0, X_1) = (Y_0, Y_1)\).

4. Interpolation spaces **X** and **Y** with respect to \((X_0, X_1)\) and \((Y_0, Y_1)\) are called of exponent \(\theta \in [0, 1]\), if there is some \(C > 0\) such that

\[ \|T\|_{\mathcal{L}(X,Y)} \leq C\|T\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta}\|T\|_{\mathcal{L}(X_1,Y_1)}^\theta \]  

(1.1)

for all \(T \in \mathcal{L}(X_j, Y_j), j = 0, 1\). If (1.1) holds with \(C = 1\), then **X** and **Y** are called exact of exponent \(\theta\). Analogously, **X** is an (exact) interpolation space of exponent \(\theta\) if the previous conditions hold for \(X = Y, (X_0, X_1) = (Y_0, Y_1)\).

**Examples 1.4**

1. Let \(0 < \alpha < 1, k \in \mathbb{N}_0\) and let

\[
\begin{align*}
C^0(\mathbb{R}^n) &= \{ f: \mathbb{R}^n \rightarrow K \text{ continuous and bounded} \} \\
\| f \|_{C^0(\mathbb{R}^n)} &= \sup_{x \in \mathbb{R}^n} |f(x)| \\
C^k(\mathbb{R}^n) &= \{ f: \mathbb{R}^n \rightarrow K \text{ } k\text{-times cont. diff.}: \partial_x^\alpha f \in C^0(\mathbb{R}^n) \forall |\alpha| \leq k \} \\
\| f \|_{C^k(\mathbb{R}^n)} &= \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial_x^\alpha f(x)| \\
C^\alpha(\mathbb{R}^n) &= \{ f: \mathbb{R}^n \rightarrow K : \| f \|_{C^\alpha(\mathbb{R}^n)} < \infty \} \\
\| f \|_{C^\alpha(\mathbb{R}^n)} &= \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}
\end{align*}
\]

Then \(C^\alpha(\mathbb{R}^n)\) is an intermediate space of \(C^0(\mathbb{R}^n)\) and \(C^1(\mathbb{R}^n)\). More precisely, there is some \(C > 0\) such that

\[ \| f \|_{C^\alpha(\mathbb{R}^n)} \leq C\| f \|_{C^0(\mathbb{R}^n)}^{1-\alpha}\| f \|_{C^1(\mathbb{R}^n)}^\alpha \]  

for all \(f \in C^1(\mathbb{R}^n)\).
2. For $1 \leq p \leq \infty$ let $L^p(U, \mu)$ denote the usual Lebesgue space on a measure space $(U, \mu)$. Then for any $1 \leq p_0, p_1, p \leq \infty$ such that \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) for some $\theta \in [0, 1]$ the space $L^p(U, \mu)$ is an intermediate space of $L^{p_0}(U, \mu)$ and $L^{p_1}(U, \mu)$. This follows from the generalized Hölder inequality

\[
\|f\|_{L^p(U, \mu)} \leq \|f\|^{1-\theta}_{L^{p_0}(U, \mu)} \|f\|^\theta_{L^{p_1}(U, \mu)} \quad \text{for all } f \in L^{p_0}(U, \mu) \cap L^{p_1}(U, \mu),
\]

which itself follows easily from the standard Hölder inequality.

**Remark 1.5** The last two inequalities are special cases of (1.1) if one chooses $Y_j = \mathbb{K}$. The following theorem shows that $L^p(U, \mu)$ in the latter example is an exact interpolation space with respect to $(L^{p_0}(U, \mu), L^{p_1}(U, \mu))$ of exponent $\theta$.

**THEOREM 1.6 (Riesz-Thorin)**

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $\theta \in [0, 1]$, let $(U, \mu)$ and $(V, \nu)$ be $\sigma$-finite measure spaces and let $\mathbb{K} = \mathbb{C}$. Then

\[
T \in \mathcal{L}(L^{p_j}(U, \mu), L^{q_j}(V, \nu)), j = 0, 1 \Rightarrow T|_{L^p(U, \mu)} \in \mathcal{L}(L^p(U, \mu), L^q(V, \nu)),
\]

where

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.
\]

Moreover, for every $T \in \mathcal{L}(L^{p_j}(U, \mu), L^{q_j}(V, \nu)), j = 0, 1$,

\[
\|T\|_{\mathcal{L}(L^p, L^q)} \leq \|T\|^{1-\theta}_{\mathcal{L}(L^{p_0}, L^{q_0})} \|T\|^\theta_{\mathcal{L}(L^{p_1}, L^{q_1})}.
\]

The theorem will follow from the results of Section 2 below.

**Remark 1.7** The proof of the Riesz-Thorin interpolation theorem provided the fundamental ideas for the so-called complex interpolation method. Generally an interpolation method is an functor $\mathcal{F}$ from the category of admissible couples of Banach spaces to the category of Banach spaces such that any pair of admissible couples $X = (X_0, X_1)$, $Y = (Y_0, Y_1)$ is mapped to an interpolation space $\mathcal{F}(X, Y)$ with respect to $X, Y$.

\(^1\text{Meaning that all functions will be complex valued in the following.}\)

\(^2\text{Readers not familiar with the basic notion of category do not have to understand this precisely since no category theory will be needed in the following.}\)
In some situations the following theorem due to Marcinkiewicz can be applied, when an application of the Riesz-Thorin theorem fails.

**THEOREM 1.8 (Marcinkiewicz)**

Let \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \) with \( q_0 \neq q_1 \), let \( \theta \in (0, 1) \), and let \((U, \mu)\) and \((V, \nu)\) be measure spaces. Moreover, let

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}
\]

and assume that \( p \leq q \). Then

\[
T \in \mathcal{L}(L^{p_j}(U, \mu), L^{q_j}(V, \nu)), j = 0, 1 \Rightarrow T|_{L^p(U, \mu)} \in \mathcal{L}(L^p(U, \mu), L^q(V, \nu)),
\]

Moreover, there is some \( C_\theta \) such that for every \( T \in \mathcal{L}(L^{p_j}(U, \mu), L^{q_j}_*(V, \nu)), j = 0, 1 \),

\[
\|T\|_{\mathcal{L}(L^p, L^q)} \leq C_\theta \|T\|_{\mathcal{L}(L^{p_0}, L^{q_0}_*)}^{1 - \theta} \|T\|_{\mathcal{L}(L^{p_1}, L^{q_1}_*)}^\theta.
\]

Here \( L^{q}_*(V, \nu) \) is the weak \( L^q \)-space, defined by

\[
L^{q}_*(V, \nu) = \left\{ f : V \to \mathbb{K} \text{ measurable} : m(t; f) \leq \frac{C}{t^q} \text{ for all } t > 0 \text{ and some } C > 0 \right\}
\]

if \( 1 \leq q < \infty \), where

\[
m(t; f) = \nu(\{x \in V : |f(x)| > t\}), \quad t > 0,
\]

denotes the distribution function of \( f \). Moreover, if \( q = \infty \), then \( L^\infty_*(V, \nu) = L^\infty(V, \nu) \). We note that \( L^q_*(V, \nu) \) is quasi-normed space with quasi-norm by

\[
\|f\|_{L^q_*} = \sup_{t > 0} t m(t; f)^{\frac{1}{q}}.
\]

Here a quasi-norm \( . \) on a vector space \( X \) satisfies the same conditions as a norm except that the triangle inequality is replaced by

\[
\|x + y\| \leq C(\|x\| + \|y\|) \quad \text{for all } x, y \in X
\]

for some \( C \geq 1 \). The bounded operators and the operator norm is defined in the same way for quasi-normed spaces as for normed spaces.

**Remark:** One also denotes \( L^{q, \infty}(V, \nu) \equiv L^{q}_*(V, \nu) \).
A proof of the Marcinkiewicz interpolation theorem in the special case $p_0 = q_0 = 1, p_1 = q_1 = r$, $(U, \mu) = (\mathbb{R}^n, \lambda^n)$ can be found in the appendix. The proof provided the basic ideas for the so-called real interpolation method. The real and the complex interpolation methods are the most important methods to construct interpolation spaces and will be one of the central topic of this lecture series. We will start with the complex method since it is easier to understand in the beginning, although the real interpolation method is in some sense more flexible and universal.

2 Complex Interpolation Method

2.1 Holomorphic Functions on the Strip

For the following let $X$ be a complex Banach space and let $S = \{z \in \mathbb{C} : 0 \leq \text{Re} \, z \leq 1\}$, $S_0 = \{z \in \mathbb{C} : 0 < \text{Re} \, z < 1\}$. A mapping $f : S_0 \to X$ is called holomorphic if $z \mapsto \langle f(z), x' \rangle, z \in S_0$, is holomorphic in the usual sense for all $x' \in X'$.

**THEOREM 2.1 (Maximum Principle/Phragmen-Lindelöf Theorem)**

Let $f : S \to X$ be continuous, bounded, and holomorphic in the interior of $S$. Then

$$\sup_{z \in S} \|f(z)\|_X \leq \max \left( \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_X \right). \quad (2.1)$$

**Proof:** First let $X = \mathbb{C}$ and assume that $f(z) \to |\text{Im} \, z| \to \infty 0$. Consider the mapping $h : S \to \mathbb{C}$ with

$$h(z) = \frac{e^{i\pi z} - i}{e^{i\pi z} + i}, \quad z \in S. \quad (2.2)$$

Then $h$ is a bijective mapping from $S$ onto $U = \{z \in \mathbb{C} : |z| \leq 1\} \setminus \{\pm 1\}$, that is holomorphic in $S_0$ and maps $\partial S$ onto $\{|z| = 1\} \setminus \{\pm 1\}$. Therefore $g(z) := f(h^{-1}(z))$ is bounded and continuous on $U$ and analytic in the interior of $U$. Moreover, because of $\lim_{|\text{Im} \, z| \to \infty} f(z) = 0, \lim_{z \to \pm 1} g(z) = 0$ and we can extend $g$ to a continuous function on $\{|z| \leq 1\}$. Hence

$$|g(z)| \leq \max_{|w| = 1} |g(w)| = \max \left( \sup_{t \in \mathbb{R}} |f(it)|, \sup_{t \in \mathbb{R}} |f(1 + it)| \right),$$
which implies the statement in this case. Next, if $X = \mathbb{C}$ and $f$ is a general function as in the assumptions, then we consider $f_{\delta, z_0}(z) = e^{\delta(z-z_0)^2}f(z)$, $\delta > 0$, $z_0 \in S_0$. Since $|e^{\delta(z-z_0)^2}| \leq e^{\delta(x^2-y^2)}$ with $z - z_0 = x + iy$, $-1 \leq x \leq 1, y \in \mathbb{R}$, we have $f_{\delta, z_0}(z) \to |\text{Im } z| \to \infty 0$. Therefore

$$|f(z_0)| = |f_{\delta, z_0}(z_0)| \leq \max \left( \sup_{t \in \mathbb{R}} |f_{\delta, z_0}(it)|, \sup_{t \in \mathbb{R}} |f_{\delta, z_0}(1 + it)| \right)$$

$$\leq e^\delta \max \left( \sup_{t \in \mathbb{R}} |f(it)|, \sup_{t \in \mathbb{R}} |f(1 + it)| \right).$$

Passing to the limit $\delta \to 0$, we obtain (2.1) in the case $X = \mathbb{C}$ since $z_0 \in S$ was arbitrary.

Finally, if $f: S \to X$ is as in the assumption and $z_0 \in S_0$, we can find some $x' \in X'$ with $\|x'\|_{X'} = 1$ such that $\|f(z_0)\|_X = |\langle f(z_0), x' \rangle|$. Now one applies (2.1) to $g_{x'}(z) := \langle f(z), x' \rangle$ and obtains

$$\|f(z_0)\|_X = \|g_{x'}(z_0)\| \leq \max \left( \sup_{t \in \mathbb{R}} \|g_{x'}(it)\|, \sup_{t \in \mathbb{R}} \|g_{x'}(1 + it)\| \right)$$

$$\leq \max \left( \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_X \right).$$

for arbitrary $z_0 \in S_0$.

As a corollary we obtain the following three line theorem, which is the basis for the proof of the Riesz-Thorin interpolation theorem and the complex interpolation method.

**THEOREM 2.2 (Three Lines Theorem)** Let $f: S \to X$ be continuous, bounded, and holomorphic in the interior of $S$. Then

$$\sup_{t \in \mathbb{R}} \|f(\theta + it)\|_X \leq \left( \sup_{t \in \mathbb{R}} \|f(it)\|_X \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} \|f(1 + it)\|_X \right)^{\theta}. \quad (2.3)$$

for every $\theta \in [0, 1]$.

**Proof:** Let $M_0 = \sup_{t \in \mathbb{R}} \|f(it)\|_X$, $M_1 = \sup_{t \in \mathbb{R}} \|f(1 + it)\|_X$. Let $\lambda \in \mathbb{R}$ and define

$$F_{\lambda}(z) = e^{\lambda z}f(z).$$

Then

$$\|F_{\lambda}(z)\|_X \leq \max(M_0, e^{\lambda}M_1)$$

for arbitrary $z_0 \in S_0$. 


by Theorem 2.1. Hence
\[ \| f(\theta + it) \|_X \leq \max \left( e^{-\lambda \theta} M_0, e^{\lambda(1-\theta)} M_1 \right) \]
for all \( t \in \mathbb{R} \). Choosing \( \lambda = \ln \frac{M_0}{M_1} \) finishes the proof. \( \blacksquare \)

2.2 Complex Interpolation Method and \( L^p \)-Spaces

Let \((X_0, X_1)\) be an admissible pair of complex Banach spaces. In particular we will assume always \( \mathbb{K} = \mathbb{C} \) in the following!

**Definition 2.3** Let \( F(X_0, X_1) \) be the set of all \( f : S \to X_0 + X_1 \) that are continuous, bounded, and holomorphic in \( S_0 \) and such that
\[
t \mapsto f(it) \in C(\mathbb{R}; X_0), \quad t \mapsto f(1 + it) \in C(\mathbb{R}; X_1).
\]
We equip \( F(X_0, X_1) \) with the norm
\[
\| f \|_{F(X_0, X_1)} = \max \left( \sup_{t \in \mathbb{R}} \| f(it) \|_{X_0}, \sup_{t \in \mathbb{R}} \| f(1 + it) \|_{X_1} \right).
\]
Finally, let \( F_0(X_0, X_1) \) be the set of all \( f \in F(X_0, X_1) \) such that
\[
\lim_{|t| \to \infty} \| f(j + it) \|_{X_j} = 0 \quad \text{for} \ j = 0, 1.
\]

**Remark 2.4** Because of Theorem 2.1, we have \( F(X_0, X_1) \hookrightarrow C(S; X_0 + X_1) \).

Using this, it is not difficult to show that \( F(X_0, X_1) \) is a Banach space. Moreover, obviously \( F_0(X_0, X_1) \) is a closed subspace of \( F(X_0, X_1) \).

We will need the following technical lemma.

**Lemma 2.5** The linear hull of the functions \( e^{\delta z^2 + \lambda z} a, \ a \in X_0 \cap X_1, \ \delta > 0, \ \lambda \in \mathbb{R} \) is dense in \( F_0(X_0, X_1) \).

We refer to [1, Lemma 4.2.3] for a proof.

**Definition 2.6** Let \( \theta \in (0, 1) \). Then the complex interpolation space \((X_0, X_1)[\theta] \)
is defined as
\[
(X_0, X_1)[\theta] = \{ f(\theta) : f \in F(X_0, X_1) \}
\]
\[
\| x \|_{(X_0, X_1)[\theta]} = \inf_{f \in F(X_0, X_1) : f(\theta) = x} \| f \|_{F(X_0, X_1)}.
\]
Remarks 2.7  
1. Since \( N_\theta = \{ f \in F(X_0, X_1) : f(\theta) = 0 \} \) is closed in \( F(X_0, X_1) \), we have that \((X_0, X_1)[\theta] \cong F(X_0, X_1)/N_\theta\) is a Banach space.

2. For every \( \theta \in (0, 1) \) we have

\[
(X_0, X_1)[\theta] = (X_1, X_0)[1-\theta].
\]

3. One can replace \( F(X_0, X_1) \) by \( F_0(X_0, X_1) \) in the definition of \((X_0, X_1)[\theta]\). In order to see that this gives the same space and norm, one uses that

\[
f_\delta = e^{\delta(z-\theta)^2} f(z) \in F_0(X_0, X_1)
\]

if \( f \in F(X_0, X_1) \) and \( \delta > 0 \). Moreover, \( f_\delta(\theta) = f(\theta) \) and \( \| f_\delta \|_{F_0(X_0, X_1)} \leq \max(e^{\delta\theta^2}, e^{\delta(1-\theta)^2}) \| f \|_{F(X_0, X_1)} \).

4. We have \( X_0 \cap X_1 \hookrightarrow (X_0, X_1)[\theta] \) since we can choose \( f(z) = x \) for all \( z \in S \) and \( x \in X_0 \cap X_1 \). Moreover, \((X_0, X_1)[\theta] \hookrightarrow X_0 + X_1 \) since \( F(X_0, X_1) \hookrightarrow C(S; X_0 + X_1) \). Summing up, we have \((X_0, X_1)[\theta]\) is an intermediate space with respect to \((X_0, X_1)\), i.e.,

\[
X_0 \cap X_1 \hookrightarrow (X_0, X_1)[\theta] \hookrightarrow X_0 + X_1
\]

5. For every \( \theta \in (0, 1) \) we have \((X_0, X_0)[\theta] = X_0\) with same norm.

Because of Lemma 2.5 and statement 3 above, we obtain

Corollary 2.8 Let \( \theta \in (0, 1) \). Then \( X_0 \cap X_1 \) is dense in \((X_0, X_1)[\theta]\).

Theorem 2.9 Let \( 1 \leq p_0, p_1 \leq \infty \), \( \theta \in (0, 1) \), and let \((U, \mu)\) be a measure space. Then

\[
(L^{p_0}(U, \mu), L^{p_1}(U, \mu))[\theta] = L^p(U, \mu)
\]

with the same norm, where

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.
\]

Proof: See [2, Example 2.1.11].

---

8
2.3 Properties of Complex Interpolation Spaces

**Theorem 2.10** Let \( \theta \in (0, 1) \) and let \((X_0, X_1)\) and \((Y_0, Y_1)\) be two admissible pairs of Banach spaces. Then \((X_0, X_1)[\theta]\) and \((Y_0, Y_1)[\theta]\) are exact interpolation spaces of exponent \(\theta\) with respect to \((X_0, X_1)\) and \((Y_0, Y_1)\). In particular,

\[
\|T\|_{L(X_\theta, Y_\theta)} \leq \|T\|_{L(X_0, Y_0)^{1-\theta}} \|T\|_{L(X_1, Y_1)^{\theta}}
\]

for every \(T \in L(X_j, Y_j), j = 0, 1\), where 

\[
X_\theta = (X_0, X_1)[\theta], \quad Y_\theta = (Y_0, Y_1)[\theta].
\]

**Proof:** See [2, Theorem 2.1.6]

**Corollary 2.11** For every \(\theta \in (0, 1)\) we have

\[
\|x\|_{(X_0, X_1)[\theta]} \leq \|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}
\]

for all \(x \in X_0 \cap X_1\).

Moreover, we note that Theorem 2.10 and Theorem 2.9 imply the Riesz-Thorin Interpolation Theorem 1.6.

2.4 Extensions and Generalizations

The proof of the Theorem 2.9 can be easily modified to the case of weighted-\(L^p\)-spaces with respect to a fixed reference measure \(\mu\). More precisely:

**Theorem 2.12** Let \(1 \leq p_0, p_1 < \infty, \theta \in (0, 1)\), and let \((U, \mu)\) be a measure space. Moreover, let \(\omega_j: U \to (0, \infty), j = 0, 1\), be measurable and set

\[
\omega(x) = \omega_0(x)^{p_0/(1-\theta)}\omega_1(x)^{\theta p_1}.
\]

Then

\[
(L^{p_0}(U, \omega_0 \, d\mu), L^{p_1}(U, \omega_1 \, d\mu))[\theta] = L^p(U, \omega \, d\mu),
\]

with the same norm, where

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.
\]

Here the measure \(\omega \, d\mu\) is defined by

\[
\omega \, d\mu(A) = \int_A \omega(x) \, d\mu(x)
\]

for every measurable \(A \subseteq U\).
A useful generalization of Theorem 2.10 is:

**THEOREM 2.13** Let \((X_0, X_1), (Y_0, Y_1)\) be admissible couples of Banach spaces and let \(T_z \in \mathcal{L}(X_0 \cap X_1, Y_0 + Y_1)\), \(z \in S\), be such that \(z \mapsto T_z x\) is continuous on \(S\) and holomorphic in \(S_0\) for every \(x \in X_0 \cap X_1\). Moreover, assume that \(t \mapsto T_{it} \in C(\mathbb{R}; \mathcal{L}(X_0, Y_0))\) and \(t \mapsto T_{1+it} \in C(\mathbb{R}; \mathcal{L}(X_1, Y_1))\) (with bounded operator norms). Then for every \(\theta \in (0, 1)\)

\[
\|T_\theta x\|_{(Y_0, Y_1)[\theta]} \leq M_0^{1-\theta} M_1^\theta \|x\|_{(X_0, X_1)[\theta]} \quad \text{for all } x \in X_0 \cap X_1,
\]

where

\[
M_0 = \sup_{t \in \mathbb{R}} \|T_{it}\|_{\mathcal{L}(X_0, Y_0)}, \quad M_1 = \sup_{t \in \mathbb{R}} \|T_{1+it}\|_{\mathcal{L}(X_1, Y_1)}.
\]

In particular \(T_\theta\) extends to a bounded operator from \((X_0, X_1)[\theta]\) to \((Y_0, Y_1)[\theta]\) with operator norm bounded by \(M_0^{1-\theta} M_1^\theta\).

**Proof:** See [2, Theorem 2.1.7].
3 Real Interpolation Method

3.1 K-Method

In the following let \((X_0, X_1)\) be an admissible pair of Banach spaces.

**Definition 3.1** For \(t > 0, x \in X_0 + X_1\) let

\[
K(t, x) \equiv K(t, x; X_0, X_1) = \inf_{x \in X_0 + X_1} \|x_0\|_{X_0} + t\|x_1\|_{X_1}.
\]

**Lemma 3.2** For all \(x \in X_0 + X_1\) the mapping \(t \mapsto K(t, x)\) is non-negative, increasing and concave. Moreover,

\[
K(t, x) \leq \max \left(1, \frac{t}{s}\right) K(s, x) \quad \text{for all } t, s > 0. \tag{3.1}
\]

**Definition 3.3** For \(\theta \in (0, 1), 1 \leq p \leq \infty\) we define the real interpolation space \((X_0, X_1)_{\theta, p}\) as

\[
(X_0, X_1)_{\theta, p} := \{x \in X_0 + X_1 : \Phi_{\theta, p}(K(\cdot, x)) < \infty\}
\]

where

\[
\Phi_{\theta, p}(K(\cdot, x)) = \|t^{-\theta}K(t, x)\|_{L^p((0, \infty), \frac{dt}{t})} = \begin{cases}
\left(\int_0^\infty (t^{-\theta}K(t, x))^p \frac{dt}{t}\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\sup_{t > 0} t^{-\theta}K(t, x) & \text{if } p = \infty.
\end{cases}
\]

\((X_0, X_1)_{\theta, p}\) is normed by \(\|x\|_{\theta, p} := \Phi_{\theta, p}(K(\cdot, x))\). Moreover, for \(\theta \in (0, 1)\)

\[
(X_0, X_1)_\theta := \left\{x \in (X_0, X_1)_{\theta, \infty} : \lim_{t \to 0} t^{-\theta}K(t, x) = \lim_{t \to \infty} t^{-\theta}K(t, x) = 0\right\}
\]

denotes the continuous interpolation space of \((X_0, X_1)\).

**Remarks 3.4**

1. It is easy to check that \(\|\cdot\|_{\theta, p}\) is indeed a norm and that \((X_0, X_1)_{\theta, p}\) is a linear subspace of \(X_0 + X_1\). Moreover, \((X_0, X_1)_\theta\) is a closed subspace of \((X_0, X_1)_{\theta, \infty}\).
2. Important properties of the integral with respect to \( \frac{dt}{t} \) are:

\[
\int_0^\infty f(at) \frac{dt}{t} = \int_0^\infty f(t) \frac{dt}{t}, \quad \int_0^\infty f(t^\alpha) \frac{dt}{t} = \frac{1}{|\alpha|} \int_0^\infty f(t) \frac{dt}{t}
\]

for any \( a, \alpha \neq 0 \) and integrable function \( f \).

The main result of this section is:

**THEOREM 3.5** Let \( 1 \leq p \leq \infty, \theta \in (0, 1) \) and let \( (X_0, X_1), (Y_0, Y_1) \) be admissible pairs of Banach spaces. Then \( (X_0, X_1)_\theta,p \) and \( (Y_0, Y_1)_\theta,p \) are interpolation spaces with respect to \( (X_0, X_1) \) and \( (Y_0, Y_1) \) that are exact of exponent \( \theta \). Moreover, \( (X_0, X_1)_\theta \) and \( (Y_0, Y_1)_\theta \) are interpolation spaces with respect to \( (X_0, X_1) \) and \( (Y_0, Y_1) \) that are exact of exponent \( \theta \).

The theorem follows from the following lemmas:

**Lemma 3.6** Let \( 1 \leq p_1 \leq p_2 \leq \infty, \theta \in (0, 1) \). Then

\[
X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,p_1} \hookrightarrow (X_0, X_1)_{\theta,p_2} \hookrightarrow X_0 + X_1 \tag{3.2}
\]

with continuous embeddings. Moreover, \( (X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)_{\theta} \) for any \( 1 \leq p < \infty \). In particular, \( (X_0, X_1)_{\theta,p} \) and \( (X_0, X_1)_{\theta} \) are intermediate spaces with respect to \( (X_0, X_1) \).

**Proof:** See [2, Proposition 1.1.3].

**Lemma 3.7** Let the assumptions of Theorem 3.5 be valid and let \( X = (X_0, X_1)_{\theta,p}, Y = (Y_0, Y_1)_{\theta,p} \). Then

\[
T \in \mathcal{L}(X_j, Y_j), j = 0, 1, \quad \Rightarrow \quad T|_X \in \mathcal{L}(X,Y)
\]

and

\[
\|T\|_{\mathcal{L}(X,Y)} \leq \|T\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1,Y_1)}^\theta
\]

for all \( T \in \mathcal{L}(X_j, Y_j), j = 0, 1 \). The same is true for \( X = (X_0, X_1)_\theta \) and \( Y = (Y_0, Y_1)_\theta \).

\(^3\)The first property reflects the fact that \( f \) is a Haar measure on the topological group \(((0,\infty),\cdot)\). I.e. it is a regular Borel measure that is invariant with respect to group action \( f \mapsto f(a \cdot ) \).
Proof: See [2, Theorem 1.1.6].

Note that the constants/the operator norms of the embeddings in (3.2) depend on $(\theta, p)$.

Lemma 3.8 For any $\theta \in (0,1), 1 \leq p \leq \infty$ the normed spaces $(X_0, X_1)_{\theta,p}$ and $(X_0, X_1)_\theta$ are complete.

Proof: See [2, Proposition 1.1.3].

Some more properties are as follows:

1. Since $K(t, x; X_0, X_1) = tK(t^{-1}, x; X_1, X_0)$, we have

$$(X_0, X_1)_{\theta,p} = (X_1, X_0)_{1-\theta,p} \quad (X_0, X_1)_\theta = (X_1, X_0)_{1-\theta}$$

for all $\theta \in (0,1), 1 \leq p \leq \infty$.

2. If $X_1 \hookrightarrow X_0$ continuously, then $K(t, x) \leq \|x\|_{X_0}$ and therefore

$$c_{\theta,p}\|x\|_{\theta,p} \leq \|x\|_{X_0} + \|t^{-\theta}K(t, x; X_0, X_1)\|_{L_\theta((0,1); \mathbb{R})} \leq C_{\theta,p}\|x\|_{\theta,p}$$

for some constants $c_{\theta,p}, C_{\theta,p} > 0$ depending on $\theta \in (0,1), 1 \leq p \leq \infty$.

3. For all $\theta \in (0,1), 1 \leq p \leq \infty$ we have $(X_0, X_0)_{\theta,p} = X_0$ with equivalent norms. More precisely,

$$c_{\theta,p}\|x\|_{\theta,p} \leq \|x\|_{X_0} \leq C_{\theta,p}\|x\|_{\theta,p}$$

for some constants $C_{\theta,p}, c_{\theta,p} > 0$ depending on $\theta$ and $p$!!

Proposition 3.9 Let $X_1 \hookrightarrow X_0$ continuously. Then for every $0 < \theta_1 < \theta_2 < 1$ we have

$$(X_0, X_1)_{\theta_2,\infty} \hookrightarrow (X_0, X_1)_{\theta_1,1}.$$

Recall that, if $1 \leq p \leq \infty$ and $\Omega \subseteq \mathbb{R}^n$ is a domain, then

$$W^{1}_{p}(\Omega) = \{ f \in L^{p}(\Omega) : \nabla f \in L^{p}(\Omega) \},$$

$$\|f\|_{W^{1}_{p}(\Omega)} = \|f\|_{L^{p}(\Omega)} + \|\nabla f\|_{L^{p}(\Omega)}.$$
Here $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_n} f)$ denotes the weak or distributional derivatives of $f$. More precisely, $g_j \in L^p(\Omega)$ for $1 \leq j \leq n$ is called weak derivative of $f$ in direction $x_j$, if

$$\int_{\Omega} g_j(x) \varphi(x) \, dx = - \int_{\Omega} f(x) \partial_{x_j} \varphi(x) \, dx$$

for all smooth and compactly supported $\varphi : \Omega \to \mathbb{K}$, i.e., for all $\varphi \in C_0^\infty(\Omega)$. If the weak derivative exists, it is unique and it will be for simplicity be denoted by $\partial_{x_j} f$. Note that, if $\psi \in C_0^\infty(\mathbb{R}^n)$ and $f \in W^1_p(\mathbb{R}^n)$, then $\psi \ast f \in C^1(\mathbb{R}^n)$ and

$$\partial_{x_j} \psi \ast f(x) = - \int_{\mathbb{R}^n} \partial_{y_j} \psi(x-y) f(y) \, dy = \int_{\mathbb{R}^n} \psi(x-y) \partial_{y_j} f(y) \, dy$$

because of the definition of the weak derivative. Choosing e.g. $\psi(x) = \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x) \, dx = 1$, one obtains that $\varphi_\varepsilon \ast f \to f$ in $W^1_p(\mathbb{R}^n)$. Hence $C^\infty(\mathbb{R}^n) \cap W^1_p(\mathbb{R}^n)$ is dense in $W^1_p(\mathbb{R}^n)$.

Moreover, if $U \subseteq \mathbb{R}^n$ is an open set, then $C^\theta(U)$, $\theta \in (0,1)$, denotes the set of all bounded and Hölder continuous functions $f : U \to \mathbb{K}$ of degree $\theta$ such that

$$\|f\|_{C^\theta(U)} = \|f\|_\infty + \sup_{x \neq y, x,y \in U} \frac{|f(x) - f(y)|}{|x-y|^{\theta}}$$

**Example 3.10** For every $0 < \theta < 1$ and $1 \leq p < \infty$ we have

$$(C(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta, \infty} = C^\theta(\mathbb{R}^n),$$

$$(C(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta} = h^\theta(\mathbb{R}^n),$$

$$(L^p(\mathbb{R}^n), W^1_p(\mathbb{R}^n))_{\theta,p} = W^\theta_p(\mathbb{R}^n),$$

with equivalent norms, where $h^\theta(\mathbb{R}^n)$ is the “little Hölder space” defined by

$$h^\theta(\mathbb{R}^n) = \left\{ f \in C^\theta(\mathbb{R}^n) : \lim_{R \to 0+, x,y \in \mathbb{R}^n, 0 < |x-y| \leq R} \sup_{|x-y| \leq R} \frac{|f(x) - f(y)|}{|x-y|^{\theta}} = 0 \right\}$$

equipped with the norm of $C^\theta(\mathbb{R}^n)$ and $W^\theta_p(\mathbb{R}^n)$ is the Sobolev-Slobodetskii space defined by

$$W^\theta_p(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : [f]_{W^\theta_p(\mathbb{R}^n)} < \infty \right\},$$

$$[f]_{W^\theta_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p+n}} \, d(x,y) \right)^{\frac{1}{p}}.$$
Example 3.11 Let $\Omega \subset \mathbb{R}^n$ be a domain such that there is an extension operator $E \in \mathcal{L}(C^\theta(\Omega); C^\theta(\mathbb{R}^n))$ for all $\theta \in [0, 1]$ and $Ef|_{\Omega} = f$ on $\Omega$. Then
\[(C(\Omega), C^1(\Omega))_{\theta, \infty} = C^\theta(\Omega)\]
for all $0 < \theta < 1$.

Remark 3.12 If $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$, then $E$ defined by
\[Ef(x) = \begin{cases} f(x) & \text{if } x_n \geq 0 \\ 3f(x', -x_n) - 2f(x', -2x_n) & \text{if } x_n < 0 \end{cases}\]
where $x' = (x_1, \ldots, x_{n-1})$ satisfies the conditions of (3.11). More generally, every bounded domain $\Omega$ with $C^1$-boundary satisfies the conditions of (3.11). This can be shown by locally mapping the boundary of $\Omega$ to pieces of a half space $\mathbb{R}^n_+$ and applying the half-space extension operator together with a suitable partition of unity.

Remark 3.13 On the basis on (3.5), we will be able to prove that
\[\text{Tr}_{x_n=0} W^{1}_p(\mathbb{R}^n_+) = (L^p(\mathbb{R}^{n-1}), W^{1}_p(\mathbb{R}^{n-1}))_{1 - \frac{1}{p}, p},\]
where $1 < p < \infty$, $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$, and
\[\text{Tr}_{x_n=0} W^{1}_p(\mathbb{R}^n_+) = \{a \in L^p(\mathbb{R}^{n-1}) : a(x') = f(x', 0) \text{ for some } f \in W^{1}_p(\mathbb{R}^n_+)\}\]
equipped with the quotient norm
\[\|a\|_{\text{Tr}_{x_n=0} W^{1}_p(\mathbb{R}^n_+)} = \inf_{f \in W^{1}_p(\mathbb{R}^n_+), f|_{x_n=0}=a} \|f\|_{W^{1}_p(\mathbb{R}^n_+)}.\]
To this end we will need an equivalent characterization of the real interpolation spaces $(X_0, X_1)_{\theta, p}$, namely the so-called trace method. But before we need some preliminaries on vector-valued Sobolev spaces.

3.2 Excursion: Vector-Valued $L^p$ and Sobolev Spaces

Definition 3.14 Let $X$ be a Banach space and let $(M, \mu)$ be a measure space. Then a mapping $f: M \to X$ is called simple if $f = \sum_{k=1}^N \alpha_k \chi_{A_k}$ for some measurable $A_k \subseteq M$ and $\alpha_k \in X$. Moreover, $f: M \to X$ is called strongly
measurable if there is a sequence $f_k: M \to X$ of simple functions $f_k$, $k \in \mathbb{N}$, such that $f_k(x) \to_{k \to \infty} f(x)$ in $X$ for almost every $x \in M$. Moreover, $f$ is called Bochner integrable if there is a sequence $f_k: M \to X$, $k \in \mathbb{N}$, of simple functions such that

$$\lim_{k \to \infty} \int_M \|f_k(x) - f(x)\|_X d\mu(x) = 0. \tag{3.6}$$

If $f = \sum_{k=1}^N \alpha_k \chi_{A_k}$ is a simple function, then the $X$-valued integral is defined by

$$\int_M f(x) d\mu(x) := \sum_{k=1}^N \alpha_k \mu(A_k).$$

If $f$ is Bochner integrable and $f_k$ are simple functions such that (3.6) holds, then $(\int_M f_k(x) d\mu(x))_{k \in \mathbb{N}}$ is a Cauchy sequence in $X$ and we define

$$\int_M f(x) d\mu(x) = \lim_{k \to \infty} \int_M f_k(x) d\mu(x).$$

As in the scalar case the definition is independent of the sequence $f_k$.

A useful criterion is the following:

**Proposition 3.15** Let $f: M \to X$. Then $f$ is Bochner integrable if and only if $f$ is strongly measurable and $x \mapsto \|f(x)\|_X$ is integrable.

We refer to the book by Yosida, “Functional Analysis”, Theorem 1 in Chapter V, Section 5.

From the definitions it easily follows that, if $f: M \to X$ is Bochner integrable and $A \in \mathcal{L}(X,Y)$, then $Af: M \to Y$ is Bochner integrable (w.r.t. $Y$) and

$$A \int_M f(x) d\mu(x) = \int_M Af(x) d\mu(x).$$

In particular, if $w' \in X'$ and $f: M \to X$ is Bochner integrable, then $x \mapsto \langle f(x), w' \rangle$ is integrable and

$$\left\langle \int_M f(x) d\mu(x), w' \right\rangle = \int_M \langle f(x), w' \rangle d\mu(x).$$

Here $\langle w, w' \rangle = w'(w)$ denotes the duality product of $w \in X$ and $w \in X'$.

Similarly to the scalar case we define for $1 \leq p \leq \infty$

$$L^p(M; X) = \{ f: M \to X \text{ strongly measurable } : \|f\|_X \in L^p(M) \}$$

$$\|f\|_{L^p(M; X)} = \|\|f(\cdot)\|_X\|_{L^p(M)}$$
In the following we restrict ourselves for simplicity to an interval \((a, b)\) together with the Lebesgue measure.

**Definition 3.16** Let \(f \in L^1((a, b); X) \equiv L^1(a, b; X)\). Then \(g \in L^1(a, b; X)\) is called weak derivative if

\[
\int_a^b g(t) \varphi(t) \, dt = - \int_a^b f(t) \varphi'(t) \, dt
\]

for all \(\varphi \in C^\infty_0((a, b))\).

Note that, if \(g \in L^1(a, b; X)\) is a weak derivative of \(f \in L^1(a, b; X)\), then \(g_w(x) := \langle g(x), w \rangle\) is a weak derivative of the scalar function \(f_w(x) = \langle f(x), w \rangle\) for every \(w \in X'\). In particular, the weak derivative of \(f\) is unique if it exists. Therefore it will for simplicity be denoted by \(f'\). Finally, we set

\[
W^1_p(a, b; X) = \{ f \in L^p(a, b; X) : f \text{ has a weak derivative } f' \in L^p(a, b; X) \}
\]

\[
\|f\|_{W^1_p(a, b; X)} = \|f\|_{L^p(a, b; X)} + \|f'\|_{L^p(a, b; X)}
\]

**Lemma 3.17** Let \(-\infty < a < b < \infty\), \(X\) be a Banach space and let \(1 \leq p < \infty\). Then \(C^1([a, b]; X)\) is dense in \(W^1_p(a, b; X)\) and \(W^1_p(a, b; X) \hookrightarrow C^0([a, b]; X)\) continuously. Moreover, for every \(f \in W^1_p(a, b; X)\)

\[
f(y) - f(x) = \int_x^y f'(t) \, dt \quad \text{for all } a \leq x \leq y \leq b.
\]

Here \(C^k([a, b]; X)\), \(k \in \mathbb{N}_0\), is the Banach space of all \(k\)-times differentiable functions \(f: [a, b] \to X\) equipped with a standard norm, e.g.

\[
\|f\|_{C^k([a, b]; X)} = \sup_{x \in [a, b], j = 0, \ldots, k} \|f^{(j)}(x)\|_X
\]

**Remark 3.18** If \(1 < p < \infty\), we even have

\[
W^1_p(a, b; X) \hookrightarrow C^{1 - \frac{1}{p}}([a, b]; X).
\]

where \(C^\alpha([a, b]; X)\), \(\alpha \in [0, 1]\), is defined as in the scalar case just replacing \(|.|\) by \(\|.|_X\) in the norms, i.e.,

\[
\|f\|_{C^\alpha([a, b]; X)} = \|f\|_{C^0([a, b]; X)} + \sup_{x \neq y, x, y \in [a, b]} \frac{\|f(x) - f(y)\|_X}{|x - y|^\alpha}
\]
3.3 The Trace Method

Definition 3.19 Let \((X_0, X_1)\) be an admissible pair of Banach spaces and let \(\theta \in (0, 1), 1 \leq p \leq \infty\). Then we define \(V(p, \theta, X_1, X_0)\) as the set of all \(u : (0, \infty) \to X_0 + X_1\) such that \(u \in W^1_p(a, b; X_0 + X_1) \cap L^p(a, b; X_1)\) with \(u' \in L^p(a, b; X_0)\) for all \(0 < a < b < \infty\) and finite \(\|u\|_{V(p, \theta, X_1, X_0)}\) where

\[
\|u\|_{V(p, \theta, X_0, X_1)} = \begin{cases} 
\int_0^\infty t^{\theta \varrho} (\|u(t)\|_{X_1} + \|u'(t)\|_{X_0})^p \frac{dt}{t} \frac{1}{p} & \text{if } p < \infty, \\
\text{ess sup}_{s>0} t^{\theta} (\|u(t)\|_{X_1} + \|u'(t)\|_{X_0}) & \text{if } p = \infty.
\end{cases}
\]

Remark 3.20 If \(u \in V(p, \theta, X_1, X_0)\), then \(\lim_{t \to 0} u(t)\) exists in \(X_0 + X_1\) because of

\[
\|u(t) - u(s)\|_{X_0} \leq \int_s^t \|u'(\tau)\|_{X_0} d\tau = \int_s^t \tau^{\theta} \|u'(\tau)\|_{X_0} \tau^{1-\theta} \frac{d\tau}{\tau}
\]

\[
\leq \left(\int_s^t \tau^{\theta} \|u'\|_{X_0} d\tau\right)^{\frac{1}{p}} \left(\int_s^t \tau^{(1-\theta)p'} \frac{d\tau}{\tau}\right)^{\frac{1}{p'}}
\]

\[
\leq \|u\|_{V(p, \theta, X_0, X_1)} (F(t) - F(s))^{\frac{1}{p'}}
\]

for all \(0 < s < t < \infty\) in the case \(1 < p < \infty\), where \(t \mapsto F(t) := t^{(1-\theta)}\), \(t \in [0, \infty)\), is a continuous function. If \(p = 1\) or \(p = \infty\), one simply replaces the \(L^p(0, \infty; \frac{dt}{t})\)-norm or \(L^p(0, \infty; \frac{dt}{t})\)-norm above by the \(L^\infty(0, \infty; \frac{dt}{t})\)-norm.

THEOREM 3.21 Let \((X_0, X_1)\) be an admissible pair of Banach spaces, \(\theta \in (0, 1)\) and \(1 \leq p \leq \infty\). Then

\[(X_0, X_1)_{\theta, p} = \{x \in X_0 + X_1 : x = u(0), u \in V(p, 1-\theta, X_1, X_0)\}.
\]

Moreover, the norm defined by

\[
\|x\|_{(X_0, X_1)_{\theta, p}} = \inf \{\|u\|_{V(p, 1-\theta, X_1, X_0)} : x = u(0), u \in V(p, 1-\theta, X_1, X_0)\}
\]

is equivalent to the norm of \((X_0, X_1)_{\theta, p}\).

For the proof we need the following Hardy-type inequalities:

Lemma 3.22 Let \(\alpha > 0\) and let \(1 \leq p < \infty\). Then there is a constant \(C_{\alpha, p}\) such that

\[
\int_0^\infty t^{-\alpha p} \left(\int_0^t \varphi(s) \frac{ds}{s}\right)^p \frac{dt}{t} \leq C_{\alpha, p} \int_0^\infty t^{-\alpha p} \varphi(t)^p \frac{dt}{t} \tag{3.7}
\]

\[
\int_0^\infty t^{\alpha p} \left(\int_t^\infty \varphi(s) \frac{ds}{s}\right)^p \frac{dt}{t} \leq C_{\alpha, p} \int_0^\infty t^{\alpha p} \varphi(t)^p \frac{dt}{t} \tag{3.8}
\]
for all non-negative measurable functions $\varphi: (0, \infty) \to \mathbb{R}$.

Remarks: It can be shown that $C_{\alpha,p} = |\alpha|^{-p}$ is the optimal constant in the latter inequalities, cf. the book by Hardy et al. “Inequalities”, Theorem 330.

Choosing $\alpha = \frac{p-1}{p}$, $\varphi(t) = t|f'(t)|$ and using $|f(t)| \leq \int_{0}^{t} |f'(\tau)| \, d\tau$ in Lemma 3.22, one easily derives the classical Hardy inequality:

$$\left\| \frac{f(t)}{t} \right\|_{L^p(0,\infty)} \leq C_p \|f'\|_{L^p(0,\infty)}$$

for all $f \in C^\infty_0(0, \infty)$, where $1 < p < \infty$.

Remark 3.23 The proof of Theorem 3.21 even shows that for $x \in (X_0, X_1)_{\theta,p}$ there some let $v \in V(p, 1 - \theta, X_1, X_0)$ be such that $v(0) = x$ and

$$t \mapsto t^{2-\theta} \|v'(t)\|_{X_1} \in L^p \left(0, \infty; \frac{dt}{t} \right).$$

Moreover, there is some $C_{\theta,p} > 0$ such that

$$\|v\|_{V(p,1-\theta,X_1,X_0)} + \|t^{2-\theta}v'(t)\|_{X_1} \leq C_{\theta,p} \|x\|_{\theta,p}.$$

Corollary 3.24 Let $\theta \in (0,1)$ and let $1 \leq p < \infty$. Then $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,p}$.

Corollary 3.25 Let $1 \leq p < \infty$ and let $\mathbb{R}_+^n = \{x \in \mathbb{R}_+^n : x_n > 0\}$. Then for every $v \in W^1_p(\mathbb{R}_+^n)$ we have $\operatorname{Tr}_{x_n=0} v := v|_{x_n=0} \in W^{1-\frac{1}{p}}_p(\mathbb{R}^{n-1})$. Moreover, the mapping $\operatorname{Tr}_{x_n=0}: W^1_p(\mathbb{R}_+^n) \to W^{1-\frac{1}{p}}_p(\mathbb{R}^{n-1})$ is bounded, linear and onto.

Remark 3.26 Let

$$V_0(\infty, \theta, X_1, X_0) = \left\{ u \in V(\infty, \theta, X_1, X_0) : \lim_{t \to 0/\infty} t^\theta (\|u(t)\|_{X_1} + \|u'(t)\|_{X_0}) = 0 \right\}.$$

Then

$$(X_0, X_1)_{\theta} = \{ x \in X_0 + X_1 : x = u(0) \text{ for some } u \in V_0(\infty, 1 - \theta, X_1, X_0) \}.$$

We refer to [2, Proposition 1.13] for a proof.\(^4\)

\(^4\)The proof is correct if $X_1 \hookrightarrow X_0$ and can be modified for the general case.
3.4 Intermediate Spaces and Reiteration

**Definition 3.27** Let \( \theta \in [0, 1] \), let \((X_0, X_1)\) be an admissible pair of Banach spaces and let \( Y \) be an intermediate space with respect to \((X_0, X_1)\). Then

1. \( Y \) is said to belong to the class \( J_\theta \) if there is a constant \( C > 0 \) such that
   \[
   \|x\|_Y \leq C\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^\theta \quad \text{for all } x \in X_0 \cap X_1.
   \]
   In this case we write \( Y \in J_\theta(X_0, X_1) \).

2. \( Y \) is said to belong to the class \( K_\theta \) if there is a constant \( k > 0 \) such that
   \[
   K(t, x) \leq kt^\theta\|x\|_Y \quad \text{for all } x \in Y.
   \]
   In this case we write \( Y \in K_\theta(X_0, X_1) \).

Note that, if \( \theta \in (0, 1) \), then \( Y \in K_\theta(X_0, X_1) \) means that \( Y \hookrightarrow (X_0, X_1)_{\theta, \infty} \) continuously.

**Proposition 3.28** Let \( \theta \in (0, 1) \) and let \( Y \) be an intermediate space with respect to an admissible couple of Banach spaces \((X_0, X_1)\). Then \( Y \in J_\theta(X_0, X_1) \) if and only if \( (X_0, X_1)_{\theta, 1} \hookrightarrow Y \).

Hence we have \( Y \in J_\theta(X_0, X_1) \cap K_\theta(X_0, X_1) \) if and only if
\[
(X_0, X_1)_{\theta, 1} \hookrightarrow Y \hookrightarrow (X_0, X_1)_{\theta, \infty}.
\]
In particular, \( (X_0, X_1)_{\theta, p} \in J_\theta(X_0, X_1) \cap K_\theta(X_0, X_1) \) for every \( p \in [1, \infty] \).

**Example 3.29** \( C^k(\mathbb{R}^n) \in J_{1/2}(C^{k-1}(\mathbb{R}^n), C^{k+1}(\mathbb{R}^n)) \cap K_{1/2}(C^{k-1}(\mathbb{R}^n), C^{k+1}(\mathbb{R}^n)) \) for all \( k \in \mathbb{N} \). More generally, \( C^k(\mathbb{R}^n) \in J_{(k-m_1)/(m_2-m_1)}(C^{m_1}(\mathbb{R}^n), C^{m_2}(\mathbb{R}^n)) \cap K_{(k-m_1)/(m_2-m_1)}(C^{m_1}(\mathbb{R}^n), C^{m_2}(\mathbb{R}^n)) \) for all \( m_1, m_2, k \in \mathbb{N} \) with \( m_1 < k < m_2 \).

**Remark:** It can be shown that \( C^1(\mathbb{R}^n) \) is not an interpolation space with respect to \((C(\mathbb{R}^n), C^2(\mathbb{R}^n))\), cf. [2, Example 1.3.3]!

The main result of this section is the following fundamental theorem:

**Theorem 3.30** *(Reiteration Theorem)*

Let \((X_0, X_1)\) be an admissible pair of Banach spaces and let \( 0 \leq \theta_0, \theta_1 \leq 1 \) with \( \theta_0 \neq \theta_1 \). Moreover, let \( \theta \in (0, 1) \) and set \( \omega = (1 - \theta)\theta_0 + \theta\theta_1 \). Then:
1. If \( Y_j \in K_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then
\[
(Y_0, Y_1)_{\theta, p} \hookrightarrow (X_0, X_1)_{\omega, p} \quad \text{for all } 1 \leq p \leq \infty.
\]

2. If \( Y_j \in J_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then
\[
(X_0, X_1)_{\omega, p} \hookrightarrow (Y_0, Y_1)_{\theta, p} \quad \text{for all } 1 \leq p \leq \infty.
\]

In particular, if \( Y_j \in J_{\theta_j}(X_0, X_1) \cap K_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \), then
\[
(Y_0, Y_1)_{\theta, p} = (X_0, X_1)_{\omega, p} \quad \text{for all } 1 \leq p \leq \infty
\]
with equivalent norms.

**Remark 3.31** Since \( (X_0, X_1)_{\theta, q} \in J_{\theta}(X_0, X_1) \cap K_{\theta}(X_0, X_1) \) for all \( \theta \in (0, 1) \), \( 1 \leq q \leq \infty \), Theorem 3.30 yields
\[
((X_0, X_1)_{\theta_0, q_0}, (X_0, X_1)_{\theta_1, q_1})_{\theta, p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1, p}
\]
for all \( 0 < \theta_0 \neq \theta_1 < 1 \) and \( 1 \leq p, q_0, q_1 \leq \infty \). Moreover,
\[
((X_0, X_1)_{\theta_0, q_0}, X_1)_{\theta, p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1, p}
\]
\[
(X_0, (X_0, X_1)_{\theta_1, q_1})_{\theta, p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1, p}
\]
for all \( 0 < \theta_0 \neq \theta_1 < 1 \) and \( 1 \leq p, q_0, q_1 \leq \infty \) since \( X_j \in J_{\theta_j}(X_0, X_1) \cap K_{\theta_j}(X_0, X_1) \) for \( j = 0, 1 \).

As consequences of the previous identity we obtain
\[
(C^{\theta_0}({\mathbb{R}}^n), C^{\theta_1}({\mathbb{R}}^n))_{\theta, \infty} = (C^{\theta_0}({\mathbb{R}}^n), C^{1}({\mathbb{R}}^n))_{(1-\theta)\theta_0+\theta\theta_1, \infty} = C^{(1-\theta)\theta_0+\theta\theta_1}({\mathbb{R}}^n),
\]
\[
(W^{\theta_0}_p({\mathbb{R}}^n), W^{\theta_1}_p({\mathbb{R}}^n))_{\theta, p} = (L^p({\mathbb{R}}^n), W^{1}_p({\mathbb{R}}^n))_{(1-\theta)\theta_0+\theta\theta_1, p} = W^{(1-\theta)\theta_0+\theta\theta_1}_p({\mathbb{R}}^n)
\]
for all \( 0 \leq \theta_0 \neq \theta_1 \leq 1 \), \( \theta \in (0, 1) \), and \( 1 \leq p < \infty \) due to Example 3.10. Moreover, since
\[
C^1({\mathbb{R}}^n) \subset J_{\frac{1}{2}}(C^0({\mathbb{R}}^n), C^2({\mathbb{R}}^n)) \cap K_{\frac{1}{2}}(C^0({\mathbb{R}}^n), C^2({\mathbb{R}}^n)),
\]
cf. Example 3.29, we have
\[
(C^0({\mathbb{R}}^n), C^2({\mathbb{R}}^n))_{\omega, \infty} = (C^0({\mathbb{R}}^n), C^1({\mathbb{R}}^n))_{2\omega, \infty} = C^{2\omega}({\mathbb{R}}^n)
\]
for all \( \omega \in (0, \frac{1}{2}) \).

In order to show \( (X_0, X_1)_{[\theta]} \in K_\theta(X_0, X_1) \) we will need the following integral representation for bounded and holomorphic function on the strip:
Lemma 3.32 (Poisson formula)

Let $X$ be a Banach space and let $f : S \to X$ be continuous, bounded, and holomorphic in $S_0$. Then

$$f(z) = f_0(z) + f_1(z),$$

where

$$f_j(z) = \int_{\mathbb{R}} P_j(z, t) f(j + it) dt \quad \text{for all } z \in S_0, j = 0, 1, \quad (3.9)$$

and

$$P_j(x + iy, t) = \frac{e^{\pi(y-t)} \sin(\pi x)}{\sin^2(\pi x) + (\cos(\pi x) - (-1)^j \exp(\pi(y - t)))^2}.$$ 

Proof: (Sketch) First let $X = \mathbb{C}$, recall the Poisson formula for the unit circle:

$$f(z) = \frac{1}{2\pi} \int_{|w| = 1} f(w) \frac{1 - |z|^2}{|w - z|^2} dw \quad \text{for all } |z| < 1,$$

which can be derived from the Cauchy formula, cf. e.g. Remmert: “Funktionsentheorie I”. The representation on $S$ can be derived from the latter Poisson formula with the aid of the mapping $h : S \to \{|z| \leq 1\}$ as in (2.2). Finally, the case that $X$ is a general Banach space can be easily reduced to the scalar case by considering $z \mapsto \langle f(z), x' \rangle$ for an arbitrary $x' \in X'$.

Proposition 3.33 For every $\theta \in (0, 1)$ we have $(X_0, X_1)_{[\theta]} \in K_\theta(X_0, X_1)$.

As a consequence we obtain

THEOREM 3.34 Let $0 < \theta_0, \theta_1, \theta < 1$ with $\theta_0 \neq \theta_1$ and let $1 \leq q \leq \infty$. Then

$$(X_0, X_1)_{[\theta_0]}(X_0, X_1)_{[\theta_1]} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, q}$$

with equivalent norms.

For completeness we note that

$$(X_0, X_1)_{[\theta_0]}(X_0, X_1)_{[\theta_1]} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, [\theta]}$$

if $X_0 \cap X_1$ is dense in $X_0, X_1$, and $(X_0, X_1)_{[\theta_0]} \cap (X_0, X_1)_{[\theta_1]}$, cf. [1, Theorem 4.6.1]. Note that, if $X_1 \hookrightarrow X_0$, the latter conditions are valid.

Moreover, for all $0 < \theta_0 \neq \theta_1 < 1$, $0 < \theta < 1$, $1 \leq p_0, p_1 \leq \infty$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$(X_0, X_1)_{[\theta_0, p_0]}(X_0, X_1)_{[\theta_1, p_1]} = (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, p},$$

cf. [1, Theorem 4.7.2].
3.5 Summary of Further Useful Results

3.5.1 Lorentz Spaces and Real Interpolation

In the following let \((U, \mu)\) be a measure space. Recall that for given measurable \(f : U \to \mathbb{R}\)

\[
m(t; f) = \mu \left( \{ x \in U : |f(x)| > t \} \right), \quad t \geq 0,
\]

is the distribution function of \(f\).

**Definition 3.35** The decreasing rearrangement of \(f\) is the function \(f^* : (0, \infty) \to [0, \infty]\) defined by

\[
f^*(t) = \inf \{ \sigma > 0 : m(\sigma; f) \leq t \}.
\]

Then \(f^* : (0, \infty) \to [0, \infty)\) is decreasing and continuous from the right. A fundamental property of \(f^*\) is:

**Lemma 3.36**

\[
m(\rho; f^*) = m(\rho, f) \quad \text{for all } \rho \geq 0.
\]

**Proof:** First of all, (3.10) implies \(f^*(m(\rho; f)) \leq \rho\). Since \(f^*\) is decreasing, this implies \(m(\rho; f^*) \leq \|0, m(\rho; f)\| = m(\rho; f)\). In order to prove the converse inequality, we note that \(f^*(m(\rho; f^*)) \leq \rho\) since \(f^*\) is continuous from the right. Hence \(m(\rho; f) \leq m(\rho; f^*)\) due to (3.10). \(\blacksquare\)

Because of the latter lemma and [3, Theorem 8.16], we obtain

\[
\int_U |f(x)|^p \, d\mu(x) = p \int_0^\infty t^{p-1} m(t; f) \, dt
\]

\[
= p \int_0^\infty t^{p-1} m(t; f^*) \, dt = \int_0^\infty |f^*(t)|^p \, dt
\]

for all \(1 \leq p < \infty\) and \(f \in L^p(U, \mu)\). Moreover,

\[
\|f\|_{L^\infty(U, \mu)} = \inf \{ t > 0 : m(t; f) = 0 \} = \|f^*\|_{L^\infty(0, \infty)}.
\]

**Definition 3.37** Let \(1 \leq p, q \leq \infty\). Then the Lorentz space \(L^{p,q}(U, \mu)\) is defined as

\[
L^{p,q}(U, \mu) = \{ f : U \to \mathbb{R} \text{ measurable : } \|f\|_{L^{p,q}} < \infty \}
\]

\[
\|f\|_{L^{p,q}} = \left\| t^{\frac{1}{p'}} f^*(t) \right\|_{L^q(0, \infty; \frac{dt}{t})}.
\]
As a direct consequence of the definition we obtain
\[ L^p(U, \mu) = L^p(U, \mu), \quad L^{p,\infty}(U, \mu) = L^p_{\text{weak}}(U, \mu) \]
for all \( 1 \leq p \leq \infty \). Moreover, since \( f^* \) is monotone, one obtains
\[ L^{p,1}(U, \mu) \hookrightarrow L^{p,q_1}(U, \mu) \hookrightarrow L^{p,q_2}(U, \mu) \hookrightarrow L^{p,\infty}(U, \mu) \quad (3.12) \]
for every \( 1 \leq q_1 \leq q_2 \leq \infty \) similarly as in the proof of Lemma 3.6.

The Lorentz spaces occur naturally as real interpolation spaces of \( L^p \)-spaces:

**Theorem 3.38** Let \( 1 \leq p_0, p_1, q_0, q_1, r \leq \infty \). Then for every \( \theta \in (0,1) \) we have
\[ (L^{p_0,q_0}(U, \mu), L^{p_1,q_1}(U, \mu))_{\theta,r} = L^{p,r}(U, \mu) \]
with equivalent norms if \( p_0 \neq p_1 \), where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \). If \( p_0 = p_1 \), the same conclusion holds if additionally \( \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \).

In the case \( 1 < p_0 \neq p_1 < \infty \), the latter theorem follows the Reiteration Theorem 3.30 and the following special case:

**Theorem 3.39** For every \( \theta \in (0,1) \)
\[ (L^1(U, \mu), L^{\infty}(U, \mu))_{\theta,r} = L^{1-\theta,\infty,\theta}(U, \mu). \]

The proof is based on the identity
\[ K(t, f; L^1, L^{\infty}) = \int_0^t f^*(\tau) \, d\tau \quad \text{for all } t > 0 \]
the monotonicity of \( f^*(t) \) and the Hardy inequality. We refer to [2] for the details.

For the general case, a proof of Theorem 3.38 can be found in [1, Theorem 5.3.1].

### 3.5.2 Dual Spaces

**Theorem 3.40** Let \((X_0, X_1)\) be an admissible couple of Banach spaces such that \(X_0 \cap X_1\) is dense in \(X_0\) and \(X_1\) and let \(1 \leq p < \infty\), \(\theta \in (0,1)\). Then
\[ ((X_0, X_1)_{\theta,p})' = (X_0', X_1')_{\theta,p'}, \]
\[ ((X_0, X_1)_{\theta})' = (X_0', X_1')_{\theta,1}. \]
Proof: See [2, Theorem 1.18].

We note that, if \( X_0 \cap X_1 \) is dense in \( X_0 \) and \( X_1 \), then

\[
(X_0 \cap X_1)' = X'_0 + X'_1, \quad (X_0 + X_1)' = X'_0 \cap X'_1
\]

with same norms, cf. [1, Theorem 2.7.1]. Therefore \((X_0, X_1)_{\theta,p}'\) and \((X'_0, X'_1)_{\theta,p'}\) are intermediate spaces with respect to \((X'_0, X'_1)\) and the equalities above make sense.

**THEOREM 3.41** Let \((X_0, X_1)\) be an admissible couple of Banach spaces such that \( X_0 \cap X_1 \) is dense in \( X_0 \) and \( X_1 \) and let \( \theta \in (0,1) \). Moreover, let \( X_0 \) or \( X_1 \) be reflexive. Then

\[
((X_0, X_1)[\varrho])' = (X'_0, X'_1)[\varrho]
\]

**Proof:** See [1, Corollary 4.5.2].
4 Bessel Potential and Besov Spaces

4.1 Mikhlin Multiplier Theorem

Recall that the Fourier transformation $\mathcal{F}$ and the inverse Fourier transformation $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}[f](\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx,$$

$$\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \, d\xi,$$

where $f \in L^1(\mathbb{R}^n)$. We note that the definitions are the same for $f \in L^1(\mathbb{R}^n; X)$, where $X$ is an arbitrary Banach space. Moreover, recall that by Plancherel’s theorem $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isomorphism with inverse $\mathcal{F}^{-1}$ and that

$$\int_{\mathbb{R}^n} f(x)g(x) \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)} \, d\xi \quad (4.1)$$

for all $f, g \in L^2(\mathbb{R}^n)$. Here (4.1) holds true for $f \in L^2(\mathbb{R}^n; X)$ and $g \in L^2(\mathbb{R}^n; X')$ if the products of the functions are understood as duality product pointwise. The proof is the same as in the scalar case or one can prove (4.1) in the vector-valued case by approximating $f, g \in L^2(\mathbb{R}^n; X)$ and $g \in L^2(\mathbb{R}^n; X')$ by simple functions and applying the identity in the scalar case. Furthermore, if $f, g \in L^2(\mathbb{R}^n; H)$ and $H$ is a Hilbert space, then $\mathcal{F} : L^2(\mathbb{R}^n; H) \to L^2(\mathbb{R}^n; H)$ is an isomorphism with inverse $\mathcal{F}^{-1}$.

**THEOREM 4.1 (Mikhlin Multiplier Theorem)**

Let $H_0, H_1$ be Hilbert spaces and let $N = n + 2$. Moreover, let $m : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(H_0, H_1)$ be an $N$-times differentiable function such that

$$\|\partial^\alpha_{\xi} m(\xi)\|_{\mathcal{L}(H_0, H_1)} \leq C|\xi|^{-|\alpha|} \quad (4.2)$$

for all $\xi \neq 0$ and $|\alpha| \leq N$ and let

$$m(D_x)f = \mathcal{F}^{-1} \left[ m(\xi)\hat{f}(\xi) \right] \quad \text{for all } f \in C^\infty_0(\mathbb{R}^n; H_0).$$

Then $m(D_x)$ extends to a bounded linear operator

$$m(D_x) : L^p(\mathbb{R}^n; H_0) \to L^p(\mathbb{R}^n; H_1) \quad \text{for all } 1 < p < \infty. \quad (4.3)$$
The theorem is proved in Appendix B. We just note that (4.1) implies that, if \( H_0 = H_1 = \mathbb{C} \),
\[
\|m(D_x)f\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in L^2(\mathbb{R}^n).
\]
Hence \( m(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n)) \). In order to prove Theorem 4.1, the main step is to show that \( m(D_x) \in \mathcal{L}(L^1(\mathbb{R}^n), L^1_{weak}(\mathbb{R}^n)) \). Once this is proved, \( m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n)) \) for all \( 1 < p \leq 2 \) by the Marcinkiewicz interpolation theorem and the case \( 2 < p < \infty \) will follow by duality. Using that the Plancharel Theorem holds also for \( f \in L^2(\mathbb{R}^n; H_j) \), the proof can be generalized to the case of general Hilbert spaces \( H_0, H_1 \).

### 4.2 A Fourier-Analytic Characterization of Hölder continuity

First of all, we recall that:

1. If \( f : \mathbb{R}^n \to \mathbb{C} \) is a continuously differentiable function such that \( f \in L^1(\mathbb{R}^n) \) and \( \partial_j f \in L^1(\mathbb{R}^n) \), then
\[
\mathcal{F}[\partial_x f] = i\xi_j \mathcal{F}[f] = i\xi_j \hat{f}(\xi).
\]
\[
(4.4)
\]

2. If \( f \in L^1(\mathbb{R}^n) \) such that \( x_j f \in L^1(\mathbb{R}^n) \), then \( \hat{f}(\xi) \) is continuously partial differentiable with respect to \( \xi_j \) and
\[
\partial_{\xi_j} \hat{f}(\xi) = \mathcal{F}[-ix_j f(x)].
\]
\[
(4.5)
\]

The latter identities show the duality between differentiability of \( f \) and decay of \( \hat{f}(\xi) \) as \( |\xi| \to \infty \) as well as decay of \( f \) for large \( x \) and differentiability of \( \hat{f} \).

In the following we will use a Littlewood-Paley partition of unity \( \varphi_j(\xi) \), \( j \in \mathbb{N}_0 \) (on \( \mathbb{R}^n \)). This is a partition of unity \( \varphi_j(\xi) \), \( j \in \mathbb{N}_0 \), on \( \mathbb{R}^n \) with \( \varphi_j \in C_0^\infty(\mathbb{R}^n) \) such that
\[
\text{supp} \varphi_j \subseteq \{ \xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1} \} \quad \text{for all } j \geq 1.
\]
\[
(4.6)
\]

The partition of unity can be constructed such that \( \text{supp} \varphi_0 \subset \overline{B_2(0)} \), \( \varphi_j(\xi) = \varphi_1(2^{j+1}\xi) \) for all \( j \geq 1 \) and (4.6) holds. Then we have
\[
|\partial_{\xi_j}^\alpha \varphi_j(\xi)| \leq C\|\partial_{\xi_j}^\alpha \varphi_1\|_{L^\infty(\mathbb{R}^n)} 2^{-|\alpha|j} \quad \text{for all } \alpha \in \mathbb{N}_0^n, j \geq 1.
\]
\[
(4.7)
\]
Moreover, we note that
\[ \varphi_j(D_x)f = \mathcal{F}^{-1} \left[ \varphi_j(\xi) \hat{f}(\xi) \right] = \hat{\varphi}_j * f \]
for all \( f \in C_0^\infty(\mathbb{R}^n) \), \( j \in \mathbb{N}_0 \), where \( \hat{\varphi}_j = \mathcal{F}^{-1}[\varphi_j] \) and
\[ \hat{\varphi}_j(x) = 2^{(j-1)n} \varphi_1(2^{j-1}x) \quad \text{for all } j \in \mathbb{N} \text{ and } x \in \mathbb{R}^n. \]
For \( f \in L^\infty(\mathbb{R}^n) \) we will define \( \varphi_j(D_x)f \) by
\[ \varphi_j(D_x)f = \hat{\varphi}_j * f. \]
Furthermore, since \( \varphi_{j-1} + \varphi_j + \varphi_{j+1} \equiv 1 \) on \( \text{supp} \varphi_j \) (where \( \varphi_{-1} \equiv 0 \)), we have
\[ \varphi_j(D_x)f = (\varphi_{j-1}(D_x) + \varphi_j(D_x) + \varphi_{j+1}(D_x)) \varphi_j(D_x)f \quad (4.8) \]
for all \( f \in \mathcal{S}(\mathbb{R}^n), j \in \mathbb{N}_0 \).
Using this decomposition, we obtain the following characterization of H"older continuous functions.

**THEOREM 4.2** Let \( 0 < s < 1 \). Then \( f \in C^s(\mathbb{R}^n) \) if and only if \( f \in L^\infty(\mathbb{R}^n) \) and
\[ \| f \|_{C^s} := \sup_{j \in \mathbb{N}_0} 2^{js} \| \hat{\varphi}_j * f \|_{L^\infty(\mathbb{R}^n)} < \infty. \]
Moreover, \( \| \cdot \|_{C^s} \) is an equivalent norm on \( C^s(\mathbb{R}^n) \).

**Proof:** First let \( f \in C^s(\mathbb{R}^n) \). Then
\[ \sup_{x \in \mathbb{R}^n} |f(x - y) - f(x)| \leq \| f \|_{C^s} |y|^s \]
for all \( y \in \mathbb{R}^n \). Since we have chosen \( \varphi_j \) such that \( \varphi_j(\xi) = \varphi_1(2^{1-j}\xi) \) for \( j \geq 1 \), we have \( \hat{\varphi}_j(x) = \psi_{2^{-j}}(x) = 2^{jn} \psi(2^jx) \), where \( \psi = \mathcal{F}^{-1}[\varphi_1(2\cdot)] \). This implies that
\[ \| \hat{\varphi}_j \|_{L^1(\mathbb{R}^n)} \leq C, \quad \| \nabla \hat{\varphi}_j \|_{L^1(\mathbb{R}^n)} \leq C2^j \quad \text{for all } j \in \mathbb{N}_0. \quad (4.9) \]
Moreover,
\[ \int_{\mathbb{R}^n} \hat{\varphi}_j(y)dy = \int_{\mathbb{R}^n} \psi(y)dy = \mathcal{F}[\psi](0) = \varphi_1(0) = 0 \]
for all $j \geq 1$. Hence
\[
\varphi_j(D_x)f = \int_{\mathbb{R}^n} f(x-y)\psi_{2^{-j}}(y)dy = \int_{\mathbb{R}^n} (f(x-y) - f(x))\psi_{2^{-j}}(y)dy \tag{4.10}
\]
and therefore
\[
\|\varphi_j(D_x)f\|_\infty \leq \|f\|_{C^s} \int_{\mathbb{R}^n} |y|^s\psi_{2^{-j}}(y)dy
= 2^{-js}\|f\|_{C^s} \int_{\mathbb{R}^n} |z|^s\psi(z)dz = C2^{-js}\|f\|_{C^s}
\]
for all $j \in \mathbb{N}$. The latter inequality implies $\|f\|_{C^2} \leq C\|f\|_{C^s}$ since also $\|\varphi_0(D_x)f\|_\infty \leq C\|f\|_\infty$.

Conversely, let $f \in L^\infty(\mathbb{R}^n)$ be such that $\|f\|_{C^2} < \infty$. Now, if $|y| \leq 1$,
\[
f(x-y) - f(x) = \sum_{2^j \leq |y|^{-1}} (f_j(x-y) - f_j(x)) + \sum_{2^j > |y|^{-1}} (f_j(x-y) - f_j(x)),
\]
where $f_j = \varphi_j(D_x)f$. In order to estimate the first sum, we use the mean value theorem to conclude that
\[
|f_j(x-y) - f_j(x)| \leq |y||\nabla f_j|_\infty. \tag{4.11}
\]
Moreover, since
\[
\partial_{x_k} f_j = \partial_{x_k}\varphi_{j-1}(D_x)f_j + \partial_{x_k}\varphi_j(D_x)f_j + \partial_{x_k}\varphi_{j+1}(D_x)f_j,
\]
due to (4.8) and
\[
\|\partial_{x_k}\varphi_l(D_x)g\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial_{x_k}\varphi_l\|_{L^1(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} \leq C2^l\|g\|_{L^\infty(\mathbb{R}^n)}
\]
for general $l \in \mathbb{N}_0$, $g \in L^\infty(\mathbb{R}^n)$, we obtain
\[
\sum_{2^j \leq |y|^{-1}} |f_j(x-y) - f_j(y)| \leq C \sum_{2^j \leq |y|^{-1}} |y||\nabla f_j|_\infty
\leq C|y| \sum_{2^j \leq |y|^{-1}} 2^{j(1-s)}\|f\|_{C^2} \leq C|y|^s\|f\|_{C^2}.
\]
The second sum is simply estimated by
\[
\sum_{2^j > |y|^{-1}} |f_j(x-y) - f_j(y)| \leq 2 \sum_{2^j > |y|^{-1}} \|f_j\|_\infty
\leq 2\|f\|_{C^2} \sum_{2^j > |y|^{-1}} 2^{-js} = C|y|^s\|f\|_{C^2}
\]
Altogether \( \|f\|_{C^s} \leq C\|f\|_{C^2} \).

**Remark 4.3** Because of (4.9) we get
\[
\|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \|\hat{\varphi}_j\|_{L^1(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},
\]
(4.12)
\[
\|\nabla\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \|\nabla\hat{\varphi}_j\|_{L^1(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)} \leq C 2^j\|f\|_{L^p(\mathbb{R}^n)}
\]
(4.13)
for any \( f \in L^p(\mathbb{R}^n), \ 1 \leq p \leq \infty, \ j \in \mathbb{N}_0 \).

### 4.3 Bessel Potential and Besov Spaces – Definitions and Basic Properties

In the following let \( \langle \xi \rangle := (1+|\xi|^2)^{\frac{1}{2}} \). We note that for any \( s \in \mathbb{R} \) the function \( \langle \xi \rangle^s \) is a smooth function satisfying
\[
|\partial^\alpha \langle \xi \rangle^s| \leq C_{s,\alpha} (1+|\xi|)^{s-|\alpha|}
\]
(4.14)
for all \( \alpha \in \mathbb{N}_0^n \) and some \( C_{s,\alpha} > 0 \). The latter estimate can be proved by considering \( f: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R} \) defined by \( f_s(\xi,t) = |(t,\xi)|^s \). Since \( f_s \) is smooth and homogeneous of degree \( s \) in \( (t,\xi) \), we have
\[
|\partial^\alpha f_s(\xi,t)| \leq C_{s,\alpha} (|t|+|\xi|)^{s-|\alpha|}
\]
uniformly in \( (\xi,t) \neq 0 \) and for all \( \alpha \in \mathbb{N}_0^n \).

Using (4.14) it is easy to show, that \( \langle \xi \rangle^s \hat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n) \) for all \( f \in \mathcal{S}(\mathbb{R}^n) \). By duality \( \langle \xi \rangle^s \hat{f} \in \mathcal{S}'(\mathbb{R}^n) \) for all \( f \in \mathcal{S}'(\mathbb{R}^n) \). Therefore we can define \( \langle D_x \rangle^s: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) as
\[
\langle D_x \rangle^s f = \mathcal{F}^{-1} \left[ \langle \xi \rangle^s \hat{f} \right] \quad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n).
\]

**Definition 4.4** Let \( s \in \mathbb{R} \) and let \( 1 < p < \infty \). Then the Bessel potential space \( H^s_p(\mathbb{R}^n) \) is defined by
\[
H^s_p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x \rangle^s f \in L^p(\mathbb{R}^n) \}
\]
\[
\|f\|_{H^s_p(\mathbb{R}^n)} = \|\langle D_x \rangle^s f\|_{L^p(\mathbb{R}^n)}.
\]

**Remarks 4.5**

1. If \( p = 2 \), then we have \( f \in H^s_2(\mathbb{R}^n) \) if and only if \( \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n) \) by Plancharell’s theorem.
2. By definition $\langle D \rangle^s: H^s_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is an isomorphism with inverse $\langle D \rangle^{-s}$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ and $\langle D \rangle^{-s}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s_p(\mathbb{R}^n)$ for any $s \in \mathbb{R}$, $1 < p < \infty$.

As a consequence of the Mikhlin multiplier theorem we obtain

**Theorem 4.6** Let $m \in \mathbb{N}_0$ and let $1 < p < \infty$. Then $H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$ with equivalent norms.

**Proof:** We first prove the embedding $H^m_p(\mathbb{R}^n) \hookrightarrow W^m_p(\mathbb{R}^n)$. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\partial^\beta_x f = \mathcal{F}^{-1} \left[ (i\xi)^\beta \hat{f}(\xi) \right] = \mathcal{F}^{-1} \left[ \frac{(i\xi)^\beta}{\langle \xi \rangle^m} \langle \xi \rangle^m \hat{f}(\xi) \right]$$

Hence in order to obtain

$$\| \partial^\beta_x f \|_{L^p(\mathbb{R}^n)} \leq C_p \| \langle D \rangle^m f \|_{L^p(\mathbb{R}^n)} \equiv C_p \| f \|_{H^m_p(\mathbb{R}^n)}$$

(4.15)

for $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$ we apply Theorem 4.1 to $m_\beta(\xi) = \frac{(i\xi)^\beta}{\langle \xi \rangle^m}$. Therefore we have to verify (4.2) for $m = m_\beta$. To this end, we use (4.14) and

$$|\partial^\alpha_x (i\xi)^\beta| \leq C_{\alpha,\beta} |\xi|^{|\beta|-|\alpha|}.$$  

(4.16)

Moreover, $(1 + |\xi|)^{-m-|\alpha|} \leq |\xi|^{-|\beta|-|\alpha|}$ if $|\beta| \leq m$. Therefore

$$|\partial^\alpha_x m_\beta(\xi)| \leq C_{\alpha,\beta} |\xi|^{-|\alpha|}$$

(4.17)

follows from (4.16), (4.14), and the following claim:

**Claim:** Let $s_1, s_2 \in \mathbb{R}$, $N \in \mathbb{N}$ and let $p_1, p_2: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be $N$-times continuously differentiable satisfying

$$|\partial^\alpha_x p_j(\xi)| \leq C|\xi|^{s_j-|\alpha|} \quad \text{for all } |\alpha| \leq N, j = 1, 2.$$  

Then

$$|\partial^\alpha_x (p_1(\xi)p_2(\xi))| \leq C' |\xi|^{s_1+s_2-|\alpha|}$$

(4.18)

for all $|\alpha| \leq N$.

**Proof of Claim:** The claim follows directly from the Leibniz’s formula.

Because of (4.17), the conditions of the Mikhlin Multiplier Theorem 4.1 are satisfied and (4.15) follows for all $|\beta| \leq m$, which proves $H^m_p(\mathbb{R}^n) \hookrightarrow W^m_p(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m_p(\mathbb{R}^n)$.  

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Hence it remains to prove \( W_p^m(\mathbb{R}^n) \hookrightarrow H_p^m(\mathbb{R}^n) \). If \( m = 2k, \ k \in \mathbb{N}_0 \), is even, then \( \langle \xi \rangle^m = (1 + |\xi|^2)^k \) is a polynomial of degree \( m \). Therefore \( \langle D_x \rangle^m \) is a differential operator of order \( m \) and

\[
\|\langle D_x \rangle^m f\|_{L^p(\mathbb{R}^n; H)} \leq C \sum_{|\alpha| \leq m} \|\partial_\alpha f\|_{L^p(\mathbb{R}^n; H)},
\]

which proves the embedding in this case.

If \( m = 2k + 1, \ k \in \mathbb{N}_0 \), is odd, then

\[
\langle \xi \rangle^m = \langle \xi \rangle^m \left( \frac{1}{\langle \xi \rangle^2} + \sum_{j=1}^n \frac{\xi_j^2}{\langle \xi \rangle^2} \right) = \frac{1}{\langle \xi \rangle^2} \langle \xi \rangle^{2k} + \sum_{j=1}^n \frac{\xi_j}{\langle \xi \rangle} \langle \xi \rangle^{2k} \xi_j,
\]

where \( \langle \xi \rangle^{2k} \) and \( \langle \xi \rangle^{2k} \xi_j \) are polynomials of degree at most \( 2k + 1 \). Hence

\[
\|\langle D_x \rangle^m f\|_{L^p(\mathbb{R}^n; H)} \leq C \sum_{|\alpha| \leq m} \sum_{j=0}^n \|m_j(D_x)\partial_\alpha f\|_{L^p(\mathbb{R}^n; H)},
\]

where \( m_0(\xi) = \langle \xi \rangle^{-1} \) and \( m_j(\xi) = \xi_j \langle \xi \rangle^{-1}, \ j = 1, \ldots, n \). Hence it remains to verify the Mikhlin condition (4.2) for \( m_j(\xi) \). If \( j = 0 \), then (4.2) for \( m(\xi) = m_0(\xi) \) follows from (4.14) with \( s = -1 \) because of \( \langle \xi \rangle^{-1-|\alpha|} \leq |\xi|^{-|\alpha|} \). If \( j = 1, \ldots, n \), then (4.2) follows for \( m(\xi) = m_j(\xi) \) from (4.14) with \( s = -1 \), (4.16) with \( \beta = e_j \), and (4.18).

Theorem 4.2 gives a motivation for the following definition of the Besov space \( B^s_{pq}(\mathbb{R}^n) \).

**Definition 4.7** Let \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \). Then

\[
B^s_{pq}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{pq}(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\|f\|_{B^s_{pq}(\mathbb{R}^n)} = \left( \sum_{j=0}^\infty 2^{jsq} \|\varphi_j(D_x) f\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \text{ if } q < \infty,
\]

\[
\|f\|_{B^s_{pq}(\mathbb{R}^n)} = \sup_{j \in \mathbb{N}_0} 2^{js} \|\varphi_j(D_x) f\|_{L^p(\mathbb{R}^n)} \text{ if } q = \infty.
\]

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Remarks 4.8  1. Because of Theorem 4.2, $C^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$ for $0 < s < 1$. More generally, $C^s_*(\mathbb{R}^n) := B^s_\infty(\mathbb{R}^n)$, $s > 0$, are called H"older-Zygmund spaces.

2. Note that $f \in B^s_{pq}(\mathbb{R}^n)$ if and only if

$$(\varphi_j(D_x)f)_{j \in \mathbb{N}_0} \in \ell^s_q(\mathbb{N}_0; L^p(\mathbb{R}^n)),$$

Here $\ell^s_q(M; X)$, $M \subseteq \mathbb{Z}$, is the space of all $X$-valued sequences $x = (x_j)_{j \in M}$ such that

$$\|x\|_{\ell^s_q(M; X)} = \begin{cases} \left( \sum_{j \in M} (2^{js}\|x_j\|_X)^q \right)^{1/q} & \text{if } q < \infty, \\ \sup_{j \in M} 2^{js}\|x_j\|_X & \text{if } q = \infty. \end{cases}$$

Moreover, we set $\ell^s_q(M; X) = \ell^0_q(M; X)$. Of course $(x_j)_{j \in M} \in \ell^s_q(M; X)$ if and only if $(2^{js}x_j)_{j \in M} \in \ell^q(M; X)$.

3. Using Plancherel’s theorem, it is not difficult to show that

$B^s_{22}(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$.

The proof is left to the reader as an exercise. But the statement will also follow from Corollary 4.15 below.

Some simple relations between the Besov spaces are summarized in the following:

Lemma 4.9 Let $s \in \mathbb{R}$, $1 \leq p, q_1, q_2 \leq \infty$, and let $\varepsilon > 0$. Then

$B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{pq_2}(\mathbb{R}^n)$ if $q_1 \leq q_2$, $B^{s+\varepsilon}_{pq_\infty}(\mathbb{R}^n) \hookrightarrow B^{s}_{pq_1}(\mathbb{R}^n)$.

Proof: The first embedding follows from

$$\ell^{q_1}(\mathbb{N}_0; X) \hookrightarrow \ell^{q_2}(\mathbb{N}_0; X), \ell^s_{q_1}(\mathbb{N}_0; X) \hookrightarrow \ell^s_{q_2}(\mathbb{N}_0; X) \quad \text{if } q_1 \leq q_2. \quad (4.19)$$

The second embedding follows $\ell^{s+\varepsilon}_\infty(\mathbb{N}_0; X) \hookrightarrow \ell^s_{1}(\mathbb{N}_0; X)$ because of

$$\|(a_j)\|_{\ell^s_{q}(\mathbb{N}_0; X)} = \sum_{j=0}^{\infty} 2^{sj}\|a_j\|_X$$

$$\leq \left( \sum_{j=0}^{\infty} 2^{-\varepsilon j} \right) \sup_{j \in \mathbb{N}_0} 2^{(s+\varepsilon)j}\|a_j\|_X \leq C\varepsilon\|(a_j)\|_{\ell^{s+\varepsilon}_\infty(\mathbb{N}_0; X)}$$
Remark 4.10 The latter lemma shows that $q$ measures regularity of $f$ on a finer scale than $s$, meaning, if $s > s'$, then $B^{s}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s'}_{pq}(\mathbb{R}^n)$ with arbitrary $1 \leq q_1, q_2 \leq \infty$.

Exercise 1 Show that 
\[ B^n_{p1}(\mathbb{R}^n) \hookrightarrow C^d(\mathbb{R}^n). \]

Hint: Use that $\varphi_j(D_x)f = \tilde{\varphi}_j \ast f$, where $\|\tilde{\varphi}_j\|_{L^1(\mathbb{R}^n)} \leq C 2^{j\frac{d}{p}}$ uniformly in $j \in \mathbb{Z}$.

In order to get a sharp comparison of Besov and Bessel potential spaces we prove:

**Theorem 4.11** Let $s \in \mathbb{R}$, $1 < p < \infty$. Then there are constants $c, C > 0$ such that
\[ c \|f\|_{H^s_p(\mathbb{R}^n)} \leq \left( \sum_{j=0}^{\infty} 2^{js} |\varphi_j(D_x)f(x)|^q \right)^{1/q} \leq C \|f\|_{H^s_p(\mathbb{R}^n)} \]
for all $f \in H^s_p(\mathbb{R}^n)$.

Remark 4.12 Because of the latter equivalent norm on $H^s_p(\mathbb{R}^n)$, one defines more generally the Triebel-Lizorkin space $F^{s}_{pq}(\mathbb{R}^n)$, $s \in \mathbb{R}, 1 < p, q < \infty$, as
\[ F^{s}_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{F^{s}_{pq}(\mathbb{R}^n)} < \infty \right\}, \]
\[ \|f\|_{F^{s}_{pq}(\mathbb{R}^n)} = \left\| \left( \sum_{j=0}^{\infty} 2^{js} |\varphi_j(D_x)f(x)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \]

Hence the latter theorem shows that $H^s_p(\mathbb{R}^n) = F^{s}_{p2}(\mathbb{R}^n)$. Finally, note that
\[ \|f\|_{F^{s}_{p2}(\mathbb{R}^n)} = \|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n; \ell^2(\mathbb{N}))}. \]

Proof of Theorem 4.11: First we will show that $\|f\|_{F^{s}_{p2}(\mathbb{R}^n)} \leq C \|f\|_{H^s_p(\mathbb{R}^n)}$. To this end we define a mapping 
\[ Q : S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n; \ell^2(\mathbb{N})). \]
by

$$(Qg)(x) = (2^{js} \varphi_j(D_x) (D_x)^{-s} g(x))_{j \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0) \quad \text{for all } x \in \mathbb{R}^n.$$ 

Then

$$(Qg)(x) = F_{\xi \rightarrow x}^{-1} [m(\xi) \hat{g}(\xi)]$$

where $m(\xi) \in L(\mathbb{C}, \ell^2(\mathbb{N}_0))$ is defined by

$$m(\xi)a = (2^{js} \varphi_j(\xi) \langle \xi \rangle^{-s})_{j \in \mathbb{N}_0}a \quad \text{for all } a \in \mathbb{C}, \xi \in \mathbb{R}^n.$$ 

In order to show that $Q$ extends to a bounded operator

$$Q: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \quad \text{for all } 1 < p < \infty,$$

we verify the condition for the Mikhlin multiplier theorem 4.1:

$$\| \partial_\alpha^{\alpha} m(\xi) \|^2_{L(\mathbb{C}, \ell^2(\mathbb{N}_0))} = \sum_{j=0}^{\infty} 2^{2js} | \partial_\xi^\alpha (\varphi_j(\xi) \langle \xi \rangle^{-s}) |^2 \leq C_{\alpha,s} 2^{2js} \langle \xi \rangle^{-2s-2|\alpha|} \chi_{\text{supp } \varphi_j}(\xi) \leq C_{\alpha,s} \langle |\xi| \rangle^{-2|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$, where we have used that $2^{j-1} \leq |\xi| \leq 2^{j+1}$ on $\text{supp } \varphi_j$ if $j \geq 1$ and

$$| \partial_\xi^\alpha (\varphi_j(\xi) \langle \xi \rangle^{-s}) | \leq C_{\alpha,s} \langle |\xi| \rangle^{-s-|\alpha|}$$

uniformly in $j \in \mathbb{N}_0$, which follows from (4.7), (4.14), and the product rule. Hence (4.20) follows and therefore

$$\| f \|^2_{F_{p,2}^s(\mathbb{R}^n)} = \| Q(D_x)^s f \|^2_{L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0))} \leq C \| (D_x)^s f \|^2_{L^p(\mathbb{R}^n)} \equiv C \| f \|^2_{H^s_p(\mathbb{R}^n)}.$$ 

Note that we have shown that

$$\tilde{Q}: H^s_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0))$$

is bounded, where

$$\tilde{Q}f = (2^{-js} Q(D_x)^s f)_{j \in \mathbb{N}_0} = (\varphi_j(D_x)f)_{j \in \mathbb{N}_0}. \quad (4.21)$$

Conversely, we define a mapping

$$R: S(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \subset L^p(\mathbb{R}^n; \ell^2(\mathbb{N}_0)) \to L^p(\mathbb{R}^n)$$
by
\[(Ra)(x) = \sum_{j=0}^{\infty} 2^{-js} \tilde{\varphi}_j(D_x)(D_x)^s a_j(x) \quad \text{for all } x \in \mathbb{R}^n, a \in \mathcal{S}(\mathbb{R}^n; \ell^2(N_0)).\]

Here \(\tilde{\varphi}_j(\xi) = \varphi_{j-1}(\xi) + \varphi_j(\xi) + \varphi_{j+1}(\xi), j \in N_0, \) and \(\varphi_{-1}(\xi) = 0. - \) Note that \(\tilde{\varphi}_j(\xi)\varphi_j(\xi) = \varphi_j(\xi)\) since \(\tilde{\varphi}_j(\xi) = 1\) on \(\text{supp } \varphi_j\). Then
\[(Ra)(x) = F_{\xi \to x}^{-1} [m(\xi)\hat{a}_j(\xi)]\]
where \(m(\xi) \in \mathcal{L}(\ell^2(N_0), \mathbb{C})\) is defined by
\[m(\xi)a = \sum_{j=0}^{\infty} 2^{-js}\tilde{\varphi}_j(\xi)\{\xi\}^s a_j \quad \text{for all } (a_j)_{j \in N_0} \in \ell^2(N_0).\]

Similarly, as before
\[\|\partial^\alpha \mathcal{L}_\xi(\mathcal{S}(\ell^2(N_0), \mathbb{C}) \leq \sum_{j=0}^{\infty} 2^{-2js} \|\partial^\alpha \mathcal{L}_\xi(\tilde{\varphi}_j(\xi)\{\xi\}^s)\|^2 \leq C_q \sum_{j=0}^{\infty} 2^{-2js} 2^{s-2|\alpha|} \chi_{\{2^{j-2} \leq |\xi| \leq 2^{j+2}\}} \leq C_q |\xi|^{-2|\alpha|},\]
where we have used that for each \(\xi \in \mathbb{R}^n\) at most 5 terms in the sum above are non-zero and that \(2^{-2js} \leq C \langle \xi \rangle^{-s} \) on \(\text{supp } \tilde{\varphi}_j \subseteq \{2^{j-2} \leq |\xi| \leq 2^{j+2}\}\). Hence, applying Theorem 4.1 once more, we obtain that \(R\) extends to a bounded operator
\[R: L^p(\mathbb{R}^n; \ell^2(N_0)) \to L^p(\mathbb{R}^n) \quad \text{for all } 1 < p < \infty.\]

Now we apply \(R\) to \(a_j = 2^{js}\tilde{\varphi}_j(D_x)f, j \in N_0\). Then
\[Ra = \sum_{j=0}^{\infty} 2^{-js} \tilde{\varphi}_j(D_x)(D_x)^s 2^{js}\tilde{\varphi}_j(D_x)f = (D_x)^s f\]
since \(\sum_{j=0}^{\infty} \varphi_j(D_x)\varphi_j(D_x) = \sum_{j=0}^{\infty} \varphi_j(D_x) = I.\) Thus
\[\|f\|_{H^p_x(\mathbb{R}^n)} = \|\langle D_x \rangle^s f\|_{L^p(\mathbb{R}^n)} = \|Ra\|_{L^p(\mathbb{R}^n)} \leq C \|\langle 2^{js}\varphi_j(D_x)f \rangle_{j \in N_0}\|_{L^p(\mathbb{R}^n; \ell^2(N_0))} = C \|f\|_{F_{2^p}(\mathbb{R}^n)},\]

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which proves the lemma. Finally, we note that the previous estimates imply that
\[
\tilde{R}: \mathcal{L}^p(B^s_{\dot{p}}(\mathbb{R}^n)) \to H^s_p(\mathbb{R}^n)
\]
is bounded, where
\[
\tilde{R}(a_j)_{j \in \mathbb{N}_0} := \langle D_x \rangle^s \mathcal{R}(2^j sa_j)_{j \in \mathbb{N}_0} = \sum_{j=0}^{\infty} \tilde{\varphi}_j(D_x) a_j \quad (4.22)
\]
and therefore \( \tilde{R} \tilde{Q} = I \) on \( H^s_p(\mathbb{R}^n) \).

Remark 4.13 Note that the latter proof shows that \( H^s_p(\mathbb{R}^n) \) is a retract of \( \mathcal{L}^p(\mathbb{R}^n; \ell^s_2(\mathbb{N}_0)) \). In general:

Definition 4.14 A Banach space \( X \) is called a retract of a Banach space \( Y \) if there are bounded, linear operators \( R: Y \to X \) and \( Q: X \to Y \) such that \( RQ = \text{id}_X \).

If \( \tilde{R}, \tilde{Q} \) are defined by (4.22) and (4.21), then \( \tilde{R} \tilde{Q} = I \) on \( H^s_p(\mathbb{R}^n) \). Note that the mappings are independent of \( p \) and \( s \). Moreover, using the same mappings \( R \) and \( Q \) as in the previous proof it is easy to show that \( B^s_{\dot{p}}(\mathbb{R}^n) \) is a retract of \( \ell^q(\mathbb{N}_0; \ell^p(\mathbb{R}^n)) \).

Corollary 4.15 Let \( 1 < p < \infty \), \( s \in \mathbb{R} \). Then
\[
\begin{align*}
B^s_{\dot{p}p}(\mathbb{R}^n) &\hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow B^s_{\dot{p}2}(\mathbb{R}^n) & \text{if } 1 < p \leq 2, \quad (4.23) \\
B^s_{\dot{2}p}(\mathbb{R}^n) &\hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow B^s_{\dot{2}p}(\mathbb{R}^n) & \text{if } 2 \leq p < \infty. \quad (4.24)
\end{align*}
\]
In particular, \( H^s_2(\mathbb{R}^n) = B^s_{22}(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \).

Proof: The statements follows from Theorem 4.11, the embeddings
\[
\begin{align*}

\ell^q(\mathbb{N}_0; \mathcal{L}^p(\mathbb{R}^n)) &\hookrightarrow \mathcal{L}^p(\mathbb{R}^n; \ell^q(\mathbb{N}_0)) & \text{if } 1 \leq q \leq p \leq \infty, \quad (4.25) \\
\mathcal{L}^p(\mathbb{N}_0; \ell^q(\mathbb{R}^n)) &\hookrightarrow \ell^q(\mathbb{R}^n; \mathcal{L}^p(\mathbb{N}_0)) & \text{if } 1 \leq p \leq q \leq \infty. \quad (4.26)
\end{align*}
\]
as well as from (4.19). Here (4.25) follows from
\[
\| (f_j)_{j \in \mathbb{Z}_0} \|_{L^p(\mathbb{R}^n; \ell^q(\mathbb{Z}_0))} \leq \left( \sum_{j=0}^{\infty} \left\| f_j(\cdot) \right\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} = \left( \sum_{j=0}^{\infty} \left\| f_j(\cdot) \right\| L^p(\mathbb{R}^n) \right)^{\frac{1}{q}},
\]
where we have used Minkowski's inequality. The inequality (4.26) is proved analogously.

4.4 Interpolation of Vector-Valued \(L^p\)-Spaces, Bessel Potential, and Besov Spaces

Recall that \(H^s_p(\mathbb{R}^n)\) and \(B^s_{pq}(\mathbb{R}^n)\) are retract of \(L^p(\mathbb{R}^n; \ell^s_2(\mathbb{Z}_0))\) and \(\ell^s_q(\mathbb{Z}_0; L^p(\mathbb{R}^n))\) with same retraction and co-retractions, which are independent of \(p, s\). More precisely,
\[
\tilde{Q}: H^s_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n; \ell^s_2(\mathbb{Z}_0)), \\
\tilde{Q}: B^s_{pq}(\mathbb{R}^n) \to \ell^s_q(\mathbb{Z}_0; L^p(\mathbb{R}^n))
\]
and
\[
\tilde{R}: L^p(\mathbb{R}^n; \ell^s_2(\mathbb{Z}_0)) \to H^s_p(\mathbb{R}^n), \\
\tilde{R}: \ell^s_q(\mathbb{Z}_0; L^p(\mathbb{R}^n)) \to B^s_{pq}(\mathbb{R}^n)
\]
are bounded linear operators satisfying \(\tilde{R}\tilde{Q} = I\) where
\[
\tilde{Q}f = (\varphi_j(D_x)f)_{j \in \mathbb{Z}_0}, \quad f \in H^s_p(\mathbb{R}^n) \cup B^s_{pq}(\mathbb{R}^n).
\]
and
\[
\tilde{R}(a_j)_{j \in \mathbb{Z}_0} := \sum_{j=0}^{\infty} \varphi_j(D_x)a_j, \quad (a_j)_{j \in \mathbb{Z}_0} \in L^p(\mathbb{R}^n; \ell^s_2(\mathbb{Z}_0)) \cup \ell^s_q(\mathbb{Z}_0; L^p(\mathbb{R}^n)).
\]
Generally we have:
Proposition 4.16 Let \((X_0, X_1)\) and \((Y_0, Y_1)\) be admissible Banach spaces and let \(Q: X_0 + X_1 \to Y_0 + Y_1\), \(R: Y_0 + Y_1 \to X_0 + X_1\) be linear mappings such that \(Q \in \mathcal{L}(X_j, Y_j)\), \(R \in \mathcal{L}(Y_j, X_j)\) and \(RQx = x\) for all \(x \in X_j\) and \(j = 0, 1\). Then
\[
R(Y_0, Y_1)_{\theta, p} = (X_0, X_1)_{\theta, p} \quad R(Y_0, Y_1)[\theta] = (X_0, X_1)[\theta]
\]
with equivalent norms for all \(\theta \in (0, 1), 1 \leq p \leq \infty\), where \(R(Y_0, Y_1)_{\theta, p}\) is equipped with the quotient norm
\[
\|x\|_{R(Y_0, Y_1)_{\theta, p}} = \inf_{y \in (Y_0, Y_1)_{\theta, p}: Ry = x} \|y\|_{\theta, p}.
\]

**Proof:** The proof of the proposition is similar to the proof of Example 3.11. 

Because of the latter proposition, it is sufficient to obtain interpolation results for \(L^p(\mathbb{R}^n; \ell^s_2(N_0))\) and \(\ell^s_q(N_0; L^p(\mathbb{R}^n))\) to characterize the real and complex interpolation spaces of \(H^s_p(\mathbb{R}^n)\) and \(B^s_{pq}(\mathbb{R}^n)\).

We start with a result for the real interpolation method:

**THEOREM 4.17** Let \(X\) be a Banach spaces, let \(s_0 \neq s_1 \in \mathbb{R}, 0 < \theta < 1\), and let \(1 \leq q_0, q_1 \leq \infty\). Then for every \(1 \leq q \leq \infty\)
\[
(\ell^{s_0}_{q_0}(\mathbb{Z}; X), \ell^{s_1}_{q_1}(\mathbb{Z}; X))_{\theta, q} = \ell^s_q(\mathbb{Z}; X),

(\ell^{s_0}_{q_0}(N_0; X), \ell^{s_1}_{q_1}(N_0; X))_{\theta, q} = \ell^s_q(N_0; X)
\]
with equivalent norms, where \(s = (1 - \theta)s_0 + \theta s_1\).

As a consequence we obtain

**THEOREM 4.18** Let \(s_0 \neq s_1 \in \mathbb{R}, 0 < \theta < 1\) and let \(s = (1 - \theta)s_0 + \theta s_1\). Then for all \(1 \leq p, q_0, q_1, q \leq \infty\)
\[
(B^{s_0}_{p,q_0}(\mathbb{R}^n), B^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta, q} = B^s_{pq}(\mathbb{R}^n)
\]
with equivalent norms. Moreover, for any \(1 \leq q \leq \infty, 1 < p < \infty\) we have
\[
(H^s_p(\mathbb{R}^n), H^s_p(\mathbb{R}^n))_{\theta, q} = B^s_{pq}(\mathbb{R}^n)
\]
with equivalent norms.

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Corollary 4.22 Let $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$, and let $1 \leq q_0, q_1, p_0, p_1 < \infty$. Moreover, let $1 = \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then

\[
\begin{align*}
(\ell_{q_0}^s(\mathbb{N}_0; L^{p_0}(\mathbb{R}^n)), \ell_{q_1}^s(\mathbb{N}_0; L^{p_1}(\mathbb{R}^n)))_{[\theta]} &= \ell_q^s(\mathbb{N}_0; L^p(\mathbb{R}^n)) \\
(L^{p_0}(\mathbb{R}^n; \ell_{q_0}^s(\mathbb{N}_0)), L^{p_1}(\mathbb{R}^n; \ell_{q_1}^s(\mathbb{N}_0)))_{[\theta]} &= L^p(\mathbb{R}^n; \ell_q^s(\mathbb{N}_0))
\end{align*}
\]

with equal norms, where $s = (1-\theta)s_0 + \theta s_1$.

As an application we obtain:
THEOREM 4.23 Let $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$ and let $s = (1 - \theta)s_0 + \theta s_1$. Then
\[
(H_{p_0}^{s_0}(\mathbb{R}^n), H_{p_1}^{s_1}(\mathbb{R}^n))[\theta] = H_{p}^{s}(\mathbb{R}^n)
\]
with equal norms for any $1 < p_0, p_1 < \infty$ and $\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$. Moreover, for any $1 \leq q_0, q_1, p_0, p_1 < \infty$ we have
\[
(B_{p_0}^{s_0}(\mathbb{R}^n), B_{p_1}^{s_1}(\mathbb{R}^n))[\theta] = B_{p}^{s}(\mathbb{R}^n)
\]
with equal norms where $\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$.

4.5 Sobolev Embeddings and Traces
We start with a Sobolev-type embedding theorem for Besov and Bessel potential spaces.

THEOREM 4.24 Let $s, s_1 \in \mathbb{R}$ with $s \leq s_1$ and $1 \leq p_1 \leq p \leq \infty$ such that
\[
s - \frac{n}{p} \leq s_1 - \frac{n}{p_1}.
\]
Then
\[
B_{p_1 q_1}^{s_1}(\mathbb{R}^n) \hookrightarrow B_{p q}^{s}(\mathbb{R}^n) \quad \text{for all } 1 \leq q_1 \leq q \leq \infty, \quad (4.27)
\]
\[
H_{p_1}^{s_1}(\mathbb{R}^n) \hookrightarrow H_{p}^{s}(\mathbb{R}^n). \quad \text{if } 1 < p_1 \leq p < \infty \quad (4.28)
\]

Proof: See [1, Theorem 6.51].

Remarks on the proof: The embedding (4.27) follows from:
\[
\varphi_k(D_x)f = \tilde{\varphi}_k(D_x) \varphi_k(D_x)f
\]
with $\tilde{\varphi}_k(D_x) = \varphi_{k-1}(D_x) + \varphi_k(D_x) + \varphi_{k+1}(D_x)$, $\varphi_{-1}(D_x) := 0$ as well as
\[
\varphi_k(D_x)g = \psi_{2^{-k}} * g,
\]
which implies
\[
\|\varphi_k(D_x)g\|_{L^p(\mathbb{R}^n)} \leq \|\psi_{2^{-k}}\|_{L^q(\mathbb{R}^n)} \|g\|_{L^{p_1}(\mathbb{R}^n)} \leq C 2^{-k \frac{n}{q}} \|g\|_{L^{p_1}(\mathbb{R}^n)},
\]
where $\frac{1}{q'} = 1 - \frac{1}{q} = \frac{1}{p_1'} - \frac{1}{p'}$. Then (4.28) follows from (4.27) by a clever interpolation.
THEOREM 4.25 Let \( 1 \leq p < \infty, 1 \leq q \leq \infty, \) and let \( \text{Tr} f = f|_{x_n=0} \) for any continuous \( f \in H_p^s(\mathbb{R}^n) \cup B_{pq}^s(\mathbb{R}^n) \). Then \( \text{Tr} \) can be extended to a bounded linear operator

\[
\text{Tr}: B_{pq}^s(\mathbb{R}^n) \to B_{pq}^{s-\frac{1}{p}}(\mathbb{R}^n),
\]

\[
\text{Tr}: H_p^s(\mathbb{R}^n) \to B_{pq}^{s-\frac{1}{p}}(\mathbb{R}^n).
\]

Proof: See [1].

Remarks on the proof: If \( m = 1 \), then

\[
\text{Tr}: H_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n) \to W_p^{m-\frac{1}{p}}(\mathbb{R}^n) = B_{pq}^{m-\frac{1}{p}}(\mathbb{R}^n-1)
\]

follows from the trace method as before. Using \( a \in B_{pq}^{m-1}(\mathbb{R}^n-1) \) if and only if \( \partial^\alpha a \in B_{pq}^{1}(\mathbb{R}^n-1) \) for all \( |\alpha| \leq m-1 \), the same follows for general \( m \in \mathbb{N} \). Then the statement follows for \( s \geq 1 \) by interpolation.

In the case \( \frac{1}{p} < s < 1 \) one uses that \( B_{pq}^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R}) \). Therefore

\[
|\text{Tr} f(x')| \leq C\| f(x', \cdot) \|_{B_{pq}^{1}(\mathbb{R})}^{\frac{1}{p}}
\]

\[
\Rightarrow \| \text{Tr} f \|_{L^p(\mathbb{R}^{n-1})} \leq C\| f \|_{L^p(\mathbb{R}^{n-1};B_{pq}^{1}(\mathbb{R}))} \leq C'\| f \|_{B_{pq}^{1}(\mathbb{R}^n)}^{\frac{1}{p}}.
\]

An iterated interpolation then yields the statement of the theorem for \( \frac{1}{p} < s < 1 \).

4.6 Equivalent Norms

The following theorem is the direct generalization of Theorem 4.2 for general Besov spaces \( B_{pq}^s(\mathbb{R}^n) \) with \( 0 < s < 1 \).

THEOREM 4.26 Let \( 0 < s < 1 \) and let \( 1 \leq p, q \leq \infty \). Then there are constants \( c, C \) (depending on \( s, p, q \)) such that

\[
c\| f \|_{B_{pq}^s(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)} + \left( \int_0^\infty \frac{\omega_p(t; f)^q dt}{t^{sq}} \right)^{\frac{1}{q}} \leq C\| f \|_{B_{pq}^s(\mathbb{R}^n)} \quad (4.29)
\]

if \( q < \infty \) and

\[
c\| f \|_{B_{pq}^s(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)} + \sup_{t>0} \frac{\omega_p(t; f)^q dt}{t^{sq}} \leq C'\| f \|_{B_{pq}^s(\mathbb{R}^n)} \quad (4.30)
\]
if \( q = \infty \), where
\[
\omega_p(t; f) = \sup_{|h| \leq t} \| f(\cdot + h) - f \|_{L^p(\mathbb{R}^n)}
\]
is the \( L^p \)-modulus of continuity of \( f \).

**Remark 4.27** We refer to [1, Theorem 6.2.5] for a more general statement in the case \( s > 0 \).

**Proof of Theorem 4.26:** We will only prove the case \( q < \infty \) since the proof in the case \( q = \infty \) is a simple variant of the proof of Theorem 4.2.

First of all, since \( t \mapsto \omega_p(t; f) \) is a monotone increasing function and \( t \) is proportional to \( 2^{-j} \) on \([2^{-j+1}, 2^{-j}]\),
\[
\int_0^1 \frac{\omega_p(t; f)^q}{t^{sq}} \frac{dt}{t} \leq C \sum_{j=0}^{\infty} 2^{sqj} \omega_p(2^{-j}; f)^q
\]
and
\[
\sum_{j=0}^{\infty} 2^{sqj} \omega_p(2^{-j}; f)^q \leq C \int_0^2 \frac{\omega_p(t; f)^q}{t^{sq}} \frac{dt}{t}.
\]
Moreover,
\[
\int_1^{\infty} \frac{\omega_p(t; f)^q}{t^{sq}} \frac{dt}{t} \leq 2^q \| f \|_{L^p(\mathbb{R}^n)}^q \int_1^{\infty} t^{-sq-1} dt = C \| f \|_{L^p(\mathbb{R}^n)}^q
\]
since \( \omega_p(t; f) \leq 2 \| f \|_{L^p(\mathbb{R}^n)} \). Hence we can replace the middle term in (4.29) by
\[
\| f \|_{L^p(\mathbb{R}^n)} + \left( \sum_{j=0}^{\infty} 2^{sqj} \omega_p(2^{-j}; f) \right)^{\frac{1}{q}}.
\]
First we prove the second inequality in (4.29). For \( f \in B_{pq}^s(\mathbb{R}^n) \) we denote \( f_k = \varphi_k(D_x) f \). Then
\[
\| f_k(\cdot + h) - f_k \|_{L^p(\mathbb{R}^n)} \leq |h| \| \nabla f_k \|_{L^p(\mathbb{R}^n)}
\]
due to (4.11) and therefore
\[
\omega_p(t; f_k) \leq t \| \nabla f_k \|_{L^p(\mathbb{R}^n)} = t \| \nabla \varphi_k(D_x) \tilde{\varphi}_k(D_x) f_k \|_{L^p(\mathbb{R}^n)} \\
\leq Ct 2^k \| \tilde{\varphi}_k(D_x) f_k \|_{L^p(\mathbb{R}^n)}
\]
because of (4.13), where \( \tilde{\varphi}_k(D_x) = \varphi_{k-1}(D_x) + \varphi_k(D_x) + \varphi_{k+1}(D_x), k \in \mathbb{N}_0 \), and \( \varphi_{-1}(D_x) = 0 \). On the other hand, \( \omega_p(t, f_k) \leq 2\|f_k\|_{L^p(\mathbb{R}^n)} \) and \( f = \sum_{k=0}^{\infty} f_k \). Therefore

\[
2^{sj} \omega_p(2^{-j}; f) \leq C \left( \sum_{j=0}^{\infty} 2^{sj} \min(1, 2^{-j+k}) \|\tilde{\varphi}_k(D_x)f_k\|_{L^p(\mathbb{R}^n)} \right)
\]

\[
\leq C \left( \sum_{j=0}^{\infty} 2^{s(j-k)} \min(1, 2^{-j+k}) 2^{sk} \|\varphi_k(D_x)f_k\|_{L^p(\mathbb{R}^n)} \right).
\]

Now, defining \( a_j = C 2^{sj} \min(1, 2^{-j}), j \in \mathbb{Z}, b_j = 2^{sj} \|\varphi_k(D_x)f_j\|_{L^p(\mathbb{R}^n)} \) if \( j \geq 0 \) and \( b_j = 0 \) else, we see that \( 2^{sj} \omega_p(2^{-j}; f) \leq (a * b)_j \), where

\[
(a * b)_j = \sum_{k \in \mathbb{Z}} a_{j-k} b_k
\]

is the convolution of two sequences. Hence

\[
\left( \sum_{j=0}^{\infty} 2^{sj} \omega_p(2^{-j}; f)^q \right)^{\frac{1}{q}} \leq \|a \ast b\|_{\ell^q(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})} \|b\|_{\ell^q(\mathbb{Z})} \leq C \|f\|_{B^{spq}_{L^p(\mathbb{R}^n)}},
\]

where \( a \in \ell^1(\mathbb{Z}) \) since \( s \in (0, 1) \). Here we have used the discrete version of Young’s inequality \( \|a \ast b\|_\ell \leq \|a\|_\ell \|b\|_\ell \), which can be proved in the same way as for the usual convolution using Hölder’s inequality.

In order to prove the first inequality in (4.29), we use that

\[
\varphi_j(D_x)f = \int_{\mathbb{R}^n} (f(x - 2^{-j} z) - f(x)) \psi(z) dz,
\]

cf. (4.10). Therefore

\[
\|\varphi_j(D_x)f\|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} ||f(x - 2^{-j} z) - f||_{L^p(\mathbb{R}^n)} |\psi(z)| dz
\]

\[
\leq \int_{\mathbb{R}^n} \omega_p(2^{-j}|z|; f) |\psi(z)| dz.
\]
and

\[ \left( \sum_{j=1}^{\infty} 2^{sjq} \| \varphi_j (D_x) f \|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \]

\[ \leq \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} 2^{sjq} \omega_p (2^{-j} |z|; f)^q \right)^{\frac{1}{q}} |\psi(z)| \, dz \]

\[ \leq C \int_{\mathbb{R}^n} \left( \int_0^{\infty} \frac{\omega_p (t; f)^q \, dt}{t^{sq}} \right)^{\frac{1}{q}} |z|^s |\psi(z)| \, dz \]

\[ = C \left( \int_0^{\infty} \frac{\omega_p (t; f)^q \, dt}{t^{sq}} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} |z|^s |\psi(z)| \, dz, \]

where we can estimate \( \sum_{j=0}^{\infty} 2^{sjq} \omega_p (2^{-j} |z|; f)^q \) by the corresponding integral by the same arguments as in the beginning of the proof. Finally, \( \| \varphi_0 (D_x) f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \), which finishes the proof.
A Proof of the Marcinkiewicz Interpolation Theorem

Proof of Theorem 1.8 if \( p_0 = q_0 = 1, p_1 = q_1 = r, \) \((U, \mu) = (\mathbb{R}^n, \lambda^n)\):

We note that the assumptions imply that

\[
\lambda(t; Tf) := \{x : |Tf(x)| > t \} \leq C_q \frac{\|f\|^q}{t^q} \quad (A.1)
\]

for \( q = 1 \) and \( q = r \) if \( r < \infty \) and \( \|Tf\|_\infty \leq C_\infty \|f\|_\infty \) if \( r = \infty \).

First we consider the case \( r < \infty \). Let \( f \in L^p(\mathbb{R}^n) \) and consider the distribution function \( \lambda(t; Tf), t > 0 \), defined as above. For given \( t > 0 \) we define \( f = f_1 + f_2 \) by

\[
f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| > t, \\ 0 & \text{else.} \end{cases}
\]

Then \( f_1 \in L^1(\mathbb{R}^n) \) and \( f_2 \in L^r(\mathbb{R}^n) \) since

\[
\int_{\mathbb{R}^n} |f_1(x)| \, dx = \int_{\mathbb{R}^n} |f_1(x)|^p |f_1(x)|^{1-p} \, dx \leq C_p(t) \|f\|_p^p
\]

and similarly

\[
\int_{\mathbb{R}^n} |f_2(x)|^r \, dx \leq C_{r,p}(t) \|f\|_p^p.
\]

Now, since \( |Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)| \), we have

\[\{x : |Tf(x)| > t \} \subseteq \{x : |Tf_1(x)| > t/2 \} \cup \{x : |Tf_2(x)| > t/2 \} .\]

Therefore

\[
\lambda(t; Tf) \leq \lambda(t/2; Tf_1) + \lambda(t/2; Tf_2)
\]

\[
\leq \frac{C_1}{t/2} \int_{\mathbb{R}^n} |f_1(x)| \, dx + \frac{Cr}{r} \int_{\mathbb{R}^n} |f_2(x)|^r \, dx
\]

\[
= \frac{2C_1}{t} \int_{|f(x)| > t} |f(x)| \, dx + \frac{2^rCr}{t^r} \int_{|f(x)| \leq t} |f(x)|^r \, dx
\]

where we have used the weak type \((1,1)\) and \((r,r)\) estimate and \((A.1)\). Next we use that

\[
\int_{\mathbb{R}^n} |Tf(x)|^p \, dx = p \int_0^\infty t^{p-1} \lambda(t; Tf) \, dt,
\]

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cf. e.g. [3, Theorem 8.16]. We combine this with the estimate before. To this end we calculate

\[
\int_0^\infty t^{p-1} t^{-1} \left( \int_{\|f\| > t} |f| \, dx \right) \, dt = \int_{\mathbb{R}^n} |f| \int_0^{\|f\|} t^{p-2} \, dt \, dx
\]

\[
= \frac{1}{p-1} \int_{\mathbb{R}^n} |f| \|f\|^{p-1} \, dx
\]

since \( p > 1 \) and similarly

\[
\int_0^\infty t^{p-1} t^{-r} \left( \int_{\|f\| \leq t} |f|^r \, dx \right) \, dt = \int_{\mathbb{R}^n} |f|^r \int_0^{\|f\|} t^{p-1-r} \, dt \, dx
\]

\[
= \frac{1}{r-p} \int_{\mathbb{R}^n} |f|^r \|f\|^{p-r} \, dx
\]

since \( p < r \). Altogether

\[
\|Tf\|_p \leq C_p \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n).
\]

Finally, if \( r = \infty \), we assume for simplicity that \( C_\infty = 1 \). Otherwise replace \( T \) by \( C_\infty^{-1} T \). Then we use the same splitting of \( f \) as before, but cut at height \( t/2 \) instead of \( t \). Hence \( |Tf_2(x)| \leq \frac{t}{2} \) since \( \|T\|_{\mathcal{L}(L^\infty(\mathbb{R}^n))} \leq 1 \). Therefore

\[
\{ x : |Tf(x)| > t \} \subseteq \{ x : |Tf_1(x)| > t/2 \}
\]

and \( \lambda(t,Tf) \leq \lambda(t/2,Tf_1) \). The rest of the proof in this case is done as before having only the first term.
B Proof of the Mikhlin Multiplier Theorem

For simplicity we will first prove the theorem in the case $H_0 = H_1 = \mathbb{C}$ and therefore $m: \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(\mathbb{C}) \cong \mathbb{C}$. How to modify the proof for the general case will be discussed below.

Note that (4.1) implies that

$$
\|m(D_x)f\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in L^2(\mathbb{R}^n).
$$

Hence $m(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$. In order to prove Theorem 4.1, the main step is to show that $m(D_x) \in \mathcal{L}(L^1(\mathbb{R}^n), L^1_{\text{weak}}(\mathbb{R}^n))$, which is the content of the next lemma. Once this is proved, $m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n))$ for all $1 < p \leq 2$ by the Marcinkiewicz interpolation theorem and the case $2 < p < \infty$ will follow by duality.

Lemma B.1 Let $M \equiv m(D_x)$ be as in Theorem 4.1. Then

$$
|\{x \in \mathbb{R}^n : |Mf(x)| > t\}| \leq \frac{C\|f\|_{L^1(\mathbb{R}^n)}}{t} \quad \text{for all } t > 0, f \in L^1(\mathbb{R}^n) \quad (B.1)
$$

for some $C > 0$ independent of $t > 0$.

In order to prove Lemma B.1, an essential ingredient will be the following kernel estimate:

Proposition B.2 Let $m: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be as in Theorem 4.1. Then there is a locally integrable, continuously differentiable function $k: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that

$$
m(D_x)f(x) = \int_{\mathbb{R}^n} k(x - y)f(y) \, dy \quad \text{for all } x \not\in \text{supp } f, \quad (B.2)
$$

for all $f \in L^2(\mathbb{R}^n)$ with compact support which satisfies

$$
|k(z)| \leq C|z|^{-n} \quad |\nabla k(z)| \leq C|z|^{-n-1} \quad (B.3)
$$

for all $z \neq 0$.

We postpone the proof of Proposition B.2 to the end of this section.
**Corollary B.3** Let $Q$ be a cube and $a \in L^1(\mathbb{R}^n)$ with $\text{supp } a \subseteq Q$ and $\int_Q a(x) \, dx = 0$. Then there is a constant $C > 0$ independent of $a$ and $Q$ such that

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |m(D_x)a(x)| \, dx \leq C\|a\|_1,$$

where $\tilde{Q} = Q^{2\sqrt{n}}$ denotes the cube with same center as $Q$ and $2\sqrt{n}$ times the side-length of $Q$.

**Proof:** Let $x_0$ denote the center of $Q$. Then $x \not\in \tilde{Q} = Q^{2\sqrt{n}}$ and $y \in Q$ implies $|x - x_0| > 2|y - x_0|$. Therefore

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |m(D_x)a(x)| \, dx = \int_{\mathbb{R}^n \setminus \tilde{Q}} |k \ast a(x)| \, dx$$

$$\leq \int_Q \int_{|x - x_0| > 2|y - x_0|} |k(x - y) - k(x - x_0)| \, dx |a(y)| \, dy$$

$$\leq C \int_Q |a(y)| \, dy,$$

since $\int_Q a(x) \, dx = 0$ provided that

$$\int_{|x| > 2|y|} |k(x - y) - k(x)| \, dx \leq C \quad \text{for all } y \in \mathbb{R}^n. \quad (B.4)$$

In order to prove the latter estimate, we use

$$k(x - y) - k(x) = -\int_0^1 y \cdot \nabla k(x - ty) \, dt.$$

If $|x| > 2|y|$, then

$$|k(x - y) - k(x)| \leq \sup_{t \in [0,1]} |\nabla_x k(x - ty)||y| \leq C|x|^{-n-1}|y|$$

since $|x - ty| \geq \frac{1}{2}|x|$ for all $t \in [0,1]$. Hence

$$\int_{|x| > 2|y|} |k(x - y) - k(x)| \, dx \leq C \int_{|x| > 2|y|} |x|^{-n-1} \, dx |y| \leq C'$$

uniformly in $y \neq 0$. 

\[\square\]
Proposition B.4 Let $f \in L^1(\mathbb{R}^n)$ be continuous and let $t > 0$. Then there are cubes $Q_j$, $j \in N \subseteq \mathbb{N}_0$, with disjoint interior and parallel to the axis such that

1. $t < \frac{1}{|Q_j|} \int_{Q_j} |f(y)| dy \leq 2^n t$ for all $j \in N$. \hfill (B.5)

2. $|f(x)| \leq t$ for (almost) all $x \not\in \bigcup_{j \in N} Q_j$.

Proof: In the following $\mathcal{D}_k$, $k \in \mathbb{Z}$, denotes the set of all “dyadic cubes” with side length $2^{-k}$ meaning the collection of all (closed) cubes $Q$ with corners on neighboring points of the lattice $2^{-k}\mathbb{Z}^n$. Moreover, we set $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$.

Let $\mathcal{C}_t'$ for given $t > 0$ be the set of all $Q \in \mathcal{D}$ satisfying the condition

$$t < \frac{1}{|Q|} \int_Q |f(x)| dx$$

and let $\mathcal{C}_t$ be the subset of all $Q \in \mathcal{C}_t'$ that are maximal with respect to inclusion in $\mathcal{C}_t'$. Every $Q \in \mathcal{C}_t'$ is contained in some $Q' \in \mathcal{C}_t$ since $|Q| \leq t^{-1} \|f\|_1$ for all $Q \in \mathcal{C}_t'$. Next, if $Q \in \mathcal{C}_t \cap \mathcal{D}_k$ and $Q \subset Q' \in \mathcal{D}_{k-1}$, then by the maximality of $Q$ we have $Q' \not\in \mathcal{C}_t'$, i.e.,

$$\frac{1}{|Q'|} \int_{Q'} |f(x)| dx \leq t.$$

Moreover, since $|Q'| = 2^n |Q|$, we get

$$t < \frac{1}{|Q|} \int_Q |f(x)| dx \leq \frac{2^n}{|Q'|} \int_{Q'} |f(x)| dx \leq 2^n t$$

for all $Q \in \mathcal{C}_t$.

Hence $\mathcal{C}_t = \{Q_j : j \in N\}$, where $Q_j, j \in N \subseteq \mathbb{N}_0$, are non-overlapping and (B.5) is satisfied for all $j \in N$.

Now let $F := \mathbb{R}^n \setminus \bigcup_{j \in N} Q_j$. If $x \in F$, then $\frac{1}{|Q|} \int_Q f(y) dy \leq t$ for every $Q \in \mathcal{D}$ such that $x \in Q$. Hence, choosing a sequence of cubes $Q_k \in \mathcal{D}_k$ with $x \in Q_k$, we obtain

$$|f(x)| = \lim_{k \to \infty} \left| \frac{1}{|Q_k|} \int_{Q_k} f(y) dy \right| \leq t \quad \text{for all } x \in F.$$
Remark B.5 The latter proposition holds for a general \( f \in L^1(\mathbb{R}^n) \) if one applies Lebesgue’s differentiation theorem in the last step of the proof.

Proof of Lemma B.1: First of all, since \( C_0^\infty(\mathbb{R}^n) \) is dense in \( L^1(\mathbb{R}^n) \), it is enough to consider \( f \in C_0^\infty(\mathbb{R}^n) \). Moreover, let \( t > 0 \) be fixed and let \( Q_j, j \in N \subseteq \mathbb{N}_0 \) be the cubes due to Proposition B.4, \( \Omega = \bigcup_{j \in N} Q_j \) and \( F = \mathbb{R}^n \setminus \Omega \). We define \( g, b \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) by

\[
g(x) = \begin{cases} f(x) & \text{if } x \in F \\ \frac{1}{|Q_j|} \int_{Q_j} f(y) dy & \text{if } x \in Q_j \end{cases}
\]

and \( b(x) = f(x) - g(x) \). Note that this implies

1. \( |g(x)| \leq 2^nt \) almost everywhere in \( \mathbb{R}^n \),
2. \( b(x) = 0 \) for every \( x \in F \) and \( \int_{Q_j} b(x) \, dx = 0 \) for each \( j \in N \).

Then

\[
|\{x : |Mf(x)| > t\}| \leq |\{x : |Mg(x)| > t/2\}| + |\{x : |Mb(x)| > t/2\}|
\]

and it is sufficient to estimate each term separately.

In order to estimate \( Mg \), we use that \( |g(x)| \leq 2^nt \) for almost every \( x \in \mathbb{R}^n \), \( f(x) = g(x) \) for \( x \in F \), \( t|\Omega| \leq \|f\|_1 \), and that \( M \in \mathcal{L}(L^2(\mathbb{R}^n)) \). More precisely,

\[
|\{x : |Mg(x)| > t/2\}| \leq \frac{4}{t^2} \int |Mg(x)|^2 \, dx \leq \frac{4}{t^2} \|m\|_{\infty} \int |g(x)|^2 \, dx \leq C\|m\|_{\infty}t^{-2}\left(\int_F t|f(x)| \, dx + t^2|\Omega|\right) \leq Ct^{-1}\|f\|_1
\]

In order to estimate \( Tb \), we apply Corollary B.3 to \( b_j(x) := b(x)\chi_{Q_j}(x) \) and conclude

\[
\int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Mb_j(x)| \, dx \leq C\|b_j\|_1 \leq 2C \int_{Q_j} |f(x)| \, dx
\]

where \( \tilde{Q}_j = Q_j^{2\sqrt{n}} \). On the other hand, since \( b \in L^2(\mathbb{R}^n) \), \( \sum_{j \in N} b_j \) and therefore \( \sum_{j \in N} Mb_j \) converge in \( L^2(\mathbb{R}^n) \) to \( b \) and \( Tb \), respectively. Hence

\[
|Mb(x)| \leq \sum_{j \in N} |Mb_j(x)| \quad \text{almost everywhere}
\]
and
\[ \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Mb(x)| \, dx \leq 2C \sum_{j \in N} \int_{Q_j} |f(x)| \, dx \leq 2C \|f\|_1, \]
where \( \tilde{\Omega} = \bigcup_{j \in N} \tilde{Q}_j \). Finally,
\[ |\{x : |Mb(x)| > t/2\}| \leq |\tilde{\Omega}| + \frac{2}{t} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Mb(x)| \, dx \leq \frac{C}{t} \|f\|_1, \]
where we have used that
\[ |\tilde{\Omega}| \leq \sum_{j \in N} |\tilde{Q}_j| \leq (2\sqrt{n})^n \sum_{j \in N} |Q_j| \leq \frac{C}{t} \sum_{j \in N} \int_{Q_j} |f(x)| \, dx \leq \frac{C}{t} \|f\|_1. \]
This finishes the proof of (B.1).

**Proof of Theorem 4.1:** Since \( \|m(D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|m\|_\infty \) and because of (B.1), we can apply the Marcinkiewicz interpolation theorem to conclude \( m(D_x) \in \mathcal{L}(L^p(\mathbb{R}^n)) \) for all \( 1 < p \leq 2 \). The statement for \( 2 < p < \infty \) follows by duality since
\[
\int_{\mathbb{R}^n} m(D_x)f(x)\overline{g(x)} \, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)} \, d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)m(\xi)\overline{\hat{g}(\xi)} \, d\xi = \int_{\mathbb{R}^n} f(x)\overline{m(D_x)g(x)} \, dx
\]
for all \( f, g \in C_0^\infty(\mathbb{R}^n) \) due to (4.1), where \( \overline{m(\xi)} = \overline{m(\xi)} \) satisfies the same estimates as \( m(\xi) \). Therefore \( \overline{m(D_x)} = m(D_x)' \in \mathcal{L}(L^p(\mathbb{R}^n)) \) if \( 2 < p < \infty \) by the first part.

It remains to prove Proposition B.2. To this end we will use the so-called *Littlewood-Paley partition of unity*, which is nowadays a standard tool in the theory of function spaces and harmonic analysis. It is frequently used to analyze mapping properties of certain operators. Usually a Littlewood-Paley or dyadic partition of unity on \( \mathbb{R}^n \setminus \{0\} \) is a decomposition of unity \( \varphi_j(\xi) \), \( j \in \mathbb{Z} \) of \( \mathbb{R}^n \setminus \{0\} \) such that
\[
\text{supp} \varphi_j(\xi) \subseteq \{\xi \in \mathbb{R}^n : c2^j \leq |\xi| \leq C2^j\} \quad \text{for all } j \in \mathbb{Z}.
\]
Here $c,C > 0$ are some suitable fixed numbers often chosen to be $c = \frac{1}{2}, C = 2$, which we will assume in the following. Such a partition can be easily constructed by choosing some non-negative $\psi \in C^\infty_0(\mathbb{R}^n)$ such that $\psi(\xi) > 0$ if and only if $\frac{1}{2} < |\xi| < 2$. Then defining $\psi_j(\xi) := \psi(2^{-j}\xi)$, $j \in \mathbb{Z}$ we have

$$\Phi(\xi) = \sum_{j \in \mathbb{Z}} \psi_j(\xi) > 0 \quad \text{for all } \xi \neq 0,$$

where we note that for each $\xi \neq 0$ the sum above contains at most two non-vanishing terms. Hence

$$\varphi_j(\xi) = \Phi(\xi)^{-1}\psi_j(\xi)$$

defines a decomposition of unity with the desired properties. Moreover, in this case $\varphi_j(\xi) = \varphi_0(2^{-j}\xi)$ since $\Phi(2^{-j}\xi) = \Phi(\xi)$, which implies that

$$|\partial^\alpha_\xi \varphi_j(\xi)| \leq C\|\partial^\alpha_\xi \varphi_0\|_{L^\infty(\mathbb{R}^n)}2^{-|\alpha|j} \quad (B.6)$$

for all $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{Z}$.

The idea of the proof of Proposition B.2 is to decompose

$$m(\xi) = \sum_{j \in \mathbb{Z}} m_j(\xi), \quad \xi \neq 0,$$

where $m_j(\xi) = \varphi_j(\xi)m(\xi)$. Then

$$|\partial^\alpha_\xi m_j(\xi)| \leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta}_\xi \varphi_j(\xi)||\partial^\beta_\xi m(\xi)| \leq C2^{-j|\alpha|} \quad (B.7)$$

because of (4.2), (B.6), and since $2^{-j-1} \leq |\xi| \leq 2^{-j+1}$ on supp $\varphi_j$.

For each part $m_j(\xi)$, we have

$$m_j(D_x)f = \mathcal{F}^{-1}\left[ m_j(\xi)\hat{f}(\xi) \right] = \int_{\mathbb{R}^n} k_j(x-y)f(y)\,dy,$$

where

$$k_j(x) = \mathcal{F}^{-1}[m_j](x) \in C^0(\mathbb{R}^n)$$

since $m_j(\xi) \in L^1(\mathbb{R}^n)$. Here we have used that

$$\mathcal{F}[f * g](\xi) = \hat{f}(\xi)\hat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n, f, g \in L^1(\mathbb{R}^n).$$
Hence formally
\[ m(D_x)f = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} k_j(x - y)f(y) \, dy, \]
where it remains to show that the sum on the right-hand side converges for \( x \notin \text{supp} \, f \) and that
\[ k(z) = \sum_{j \in \mathbb{Z}} k_j(z), \quad z \neq 0, \]
converges to a function satisfying (B.3). To this end, we need some suitable uniform estimates of \( k_j(z) \). These are a consequence of the following simple lemma:

**Lemma B.6** Let \( N \in \mathbb{N}_0 \) and let \( g: \mathbb{R}^n \rightarrow \mathbb{C} \) be an \( N \)-times differentiable function, \( N \in \mathbb{N}_0 \), with compact support. Then
\[ |\mathcal{F}^{-1}[g](x)| \leq C_N |\text{supp} \, g| \|x\|^{-N} \sup_{|\beta|=N} \|\partial_\xi^\beta g\|_{L^\infty(\mathbb{R}^n)} \quad (B.8) \]
uniformly in \( x \neq 0 \) and \( g \), where \( C_N \) depends only on \( n, N \).

**Proof:** Let \( \beta \in \mathbb{N}_0^n \) with \( |\beta| = N \). Then
\[ (-ix)^\beta \mathcal{F}^{-1}[g] = \mathcal{F}^{-1}[\partial_\xi^\beta g] \]
and therefore
\[ |x^\beta| |\mathcal{F}^{-1}[g](x)| \leq C \|\partial_\xi^\beta g\|_{L^1(\mathbb{R}^n)} \leq C |\text{supp} \, g| \sup_{|\beta|=N} \|\partial_\xi^\beta g\|_{L^\infty(\mathbb{R}^n)}. \]
Since \( \beta \in \mathbb{N}_0^n \) with \( |\beta| = N \) was arbitrary,
\[ |x|^N |\mathcal{F}^{-1}[g](x)| \leq C_N |\text{supp} \, g| \sup_{|\beta|=N} \|\partial_\xi^\beta g\|_{L^\infty(\mathbb{R}^n)} \]
for all \( x \in \mathbb{R}^n \), which completes the proof. \( \blacksquare \)

**Corollary B.7** Let \( m \) be as in the assumptions of Theorem 4.1 and let \( m_j(\xi) = m(\xi)\varphi_j(\xi), \, j \in \mathbb{Z}, \) where \( \varphi_j, j \in \mathbb{Z}, \) is the dyadic decomposition of unity of \( \mathbb{R}^n \setminus \{0\} \) as above. Then \( k_j(x) := \mathcal{F}_{\xi \rightarrow x}^{-1}[m_j] \) satisfies
\[ |\partial_\xi^\beta k_j(z)| \leq C_{\alpha,n,m} 2^{j(n+|\alpha|-M)} |z|^{-M} \quad (B.9) \]
for all \( z \neq 0, \, M = 0, \ldots, n+2, \, \alpha \in \mathbb{N}_0^0 \) for some constant \( C_{\alpha,n,m} \) independent of \( j \in \mathbb{Z}, \, z \neq 0. \)
Proof: First of all,
\[ \partial^{\alpha} k_j(z) = F^{-1} [(i\xi)^{\alpha} m_j(\xi)] . \]
Using (B.7) and \( 2^{j-1} \leq |\xi| \leq 2^{j+1} \) on \( \text{supp} \varphi_j \), one easily verifies
\[ |\partial^{\beta}_\xi ((i\xi)^{\alpha} m_j(\xi))| \leq C_{\alpha,\beta} 2^{j(|\alpha|-|\beta|)} \]
for all \( |\beta| \leq n + 2 \). Hence, applying Lemma B.6 to \((i\xi)^{\alpha} m_j\) with \( N = M = 0, \ldots, n + 2 \), we obtain
\[ |\partial^{\alpha} k_j(z)| \leq C_{n,m} \text{supp} \varphi_j 2^{j(|\alpha|-M)} |z|^{-M} \leq C'_{n,m} 2^{j(n+|\alpha|-M)} |z|^{-M} , \]
which finishes the proof.

Proof of Proposition B.2: Firstly, we will show that \( \sum_{j \in \mathbb{Z}} \partial^{\alpha} k_j(z), |\alpha| \leq 1 \) converges absolutely and uniformly on every compact subset of \( \mathbb{R}^n \setminus \{0\} \) to a function \( \partial^{\alpha} k(z) \) satisfying (B.3). The main idea of the proof is to split for given \( z \neq 0 \) the sum \( \sum_{j \in \mathbb{Z}} \partial^{\alpha} k_j(z) \) into the two parts
\[ \sum_{2^j \leq |z|^{-1}} \partial^{\alpha} k_j(z) \quad \text{and} \quad \sum_{2^j > |z|^{-1}} \partial^{\alpha} k_j(z) \]
and to show convergence and the estimate (B.3) separately.

For the first sum we use (B.9) with \( |\alpha| \leq 1 \) and \( M = 0 \). Then
\[ \sum_{j \leq \text{ld} |z|^{-1}} |\partial^{\alpha} k_j(z)| \leq C \sum_{j \leq \text{ld} |z|^{-1}} 2^{j(n+|\alpha|)} \leq C' |z|^{-n-|\alpha|} \]
where \( \text{ld} \) denotes the logarithm with respect to basis 2. For the second sum we apply (B.9) with \( |\alpha| \leq 1 \) and \( M = n + |\alpha| + 1 \) and obtain
\[ \sum_{\text{ld} |z|^{-1} < j} |\partial^{\alpha} k_j(z)| \leq 2 \sum_{\text{ld} |z|^{-1} < j} 2^{-j} |z|^{-n-|\alpha|-1} \leq C' |z|^{-n-|\alpha|} . \]
Hence \( \sum_{j \in \mathbb{Z}} \partial^{\alpha} k_j(z) \) converges absolutely and uniformly on every closed subset of \( \mathbb{R}^n \setminus \{0\} \) to a function \( k(z) \) that satisfies (B.3) for all \( |\alpha| \leq 1 \).

Finally, it remains to show that \( k(z) \) satisfies (B.2). First of all,
\[ m(\xi) \hat{f}(\xi) = \sum_{j \in \mathbb{Z}} m_j(\xi) \hat{f}(\xi), \quad f \in C^\infty_0(\mathbb{R}^n) \]

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since $\sum_{j \in \mathbb{Z}} \varphi_j(\xi)$ is locally finite. Moreover, since $|m_j(\xi)|$ is uniformly bounded w.r.t. $\xi$ and $\hat{f}(\xi) \in L^2(\mathbb{R}^n)$, the sum on the right-hand side converges in $L^2(\mathbb{R}^n)$ to the left-hand side by Lebesgue's theorem on dominated convergence. Hence

$$m(D_x)f = \lim_{k \to \infty} \sum_{|j| \leq N_k} m_j(D_x)f = \lim_{k \to \infty} \sum_{|j| \leq N_k} \int_{\mathbb{R}^n} k_j(x - y)f(y)\,dy,$$

in $L^2(\mathbb{R}^n)$ and almost everywhere for every $f \in C_0^\infty(\mathbb{R}^n)$ and some $N_k \to \infty$ as $k \to \infty$ since Fourier transformation is bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Therefore it only remains to interchange the summation and integration in last term above provided that $x \not\in \text{supp } f$. But this can be done since $\sum_{j \in \mathbb{Z}} k_j(z)$ converges absolutely and uniformly on every closed subset of $\mathbb{R}^n \setminus \{0\}$ as shown above. Hence (B.2) follows.

**Comments on the proof of the vector-valued case:** In the case that $m: \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(H_0, H_1)$ satisfies the assumptions of Theorem 4.1 for general Hilbert spaces $H_0, H_1$, we still conclude immediately that

$$m(D_x): L^2(\mathbb{R}^n; H_0) \to L^2(\mathbb{R}^n; H_1)$$

is a bounded operator due Plancharel theorem.\(^5\) - The rest of the proof can be modified in a straight-forward manner to the operator-valued case since all arguments are based on direct estimates involving only the size of $f(x)$, $m(\xi)$, and $k(z)$. One just has to replace the pointwise absolute value $|.|$ by the corresponding norms, i.e., $\|.|\_{H_0}$, $\|.|_{H_1}$, and $\|.|_{\mathcal{L}(H_0, H_1)}$. Finally, we note that the Marcinkiewicz interpolation theorem holds for general $X$-valued $L^p$-spaces for general Banach spaces $X$ since its proof is only based on arguments involving the size of the function.

\(^5\)Actually, this is the only step in the proof, where it is needed that $H_0, H_1$ are Hilbert space and not general Banach spaces.
References


