Urban Policy Preference Revelation
– A Primal-Dual Approach –

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Abstract: A city is a large, complex society. It appears hopeless to try to learn anything about city residents' policy preferences. We may have to patiently wait for natural experiments to unfold, or for confidential data to become accessible, or for expensive surveys to receive research grants. Here we suggest an alternative to these costly strategies. We suggest bringing a *primal-dual perspective* to the city, by casting a lower bound on policy preferences as the solution of a suitable linear program. Data constraints may make a direct solution to this (primal) program impossible. But an indirect (dual) solution may be easy to obtain. We illustrate this (novel) approach with respect to "centrism", as residents' preference for (or against) the traditional CBD's role as employment and shopping hub.

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1 Introduction

A city is a large, complex society. It appears hopeless to try to learn anything about city residents' policy preferences. We may have to patiently wait for natural experiments to unfold, or for confidential data to become accessible, or for expensive surveys to receive research grants. Here we suggest an alternative to these costly strategies. We suggest bringing a *primal-dual perspective* to the city, by casting a lower bound on policy preferences as the solution of a suitable linear program. Data constraints may make a direct solution to this (primal) program impossible. But an indirect (dual) solution may be easy to obtain. We illustrate this (novel, no literature appears to be available on this) approach with respect to "centrism", as residents' preference for (or against) the traditional CBD's role as employment and shopping hub.

Residents' "centrism" correlates with many issues in current *urban political economy.* E.g., an owner-occupier, or a landlord, with all his properties near the center, will not just endorse city center redevelopment. Either will also reject constraints on building height (which bind near the center yet bind nowhere else). Or, a landlord with most of her properties near the periphery, much as a nearby owner-occupier, will not just vote against an urban growth boundary, or in favor of a peripheral shopping center, say. Also, either will likely also vote against a carbon tax (Holian/Kahn (2015)) or a Pigouvian toll (both of which make commutes more costly for the peripheral resident). In short, centrism proxies residents' preferences on many contested issues in urban politics.

City residents always divide into landlords and renters. The "absentee landlord" literature has all landlords live outside the city; all city residents are renters. The "public ownership" literature lets all city residents share in urban rent; all city residents are landlords. But real cities, as this short paper's subject, exhibit ownership shares in between those extremes. We replace both the absentee landlord-assumption and the public-ownership assumption with a flexible *matching* framework. We make half the city's population a tenant and the other half a landlord (i.e. owning a home but also renting out one to a single tenant). Both tenants and landlords live in any of the city's residential "rings" around the CBD. Any landlord-tenant-pair is a spatial match, its combination of locations generating its own centrism-attitude.

We then ask for the least number of centrist landlords consistent with the distribution of households (or real estate) across city rings. This is the primal "minimum centrists" problem. But landlord-tenant matchings are unobservable, and hence so are minimum centrists. Alternatively, we may ask for the largest valuation of ring populations consistent with the set of matches producing a centrist preference. This is the corresponding dual "maximum valuation" problem. The paper's contribution is to prove that the solution to either program is the *greatest cumulative ring difference.* As corollary, computing (unobservable) minimum centrists only requires computing the greatest cumulative sum of (easily observable) ring differences. Rather than extract policy preference "brute force", we should exploit observable knowledge of the built environment.

Our analysis complements, and also extends, the analysis of suburbanization set out in Dascher (2019). There the "greatest cumulative ring difference" is shown to bound the

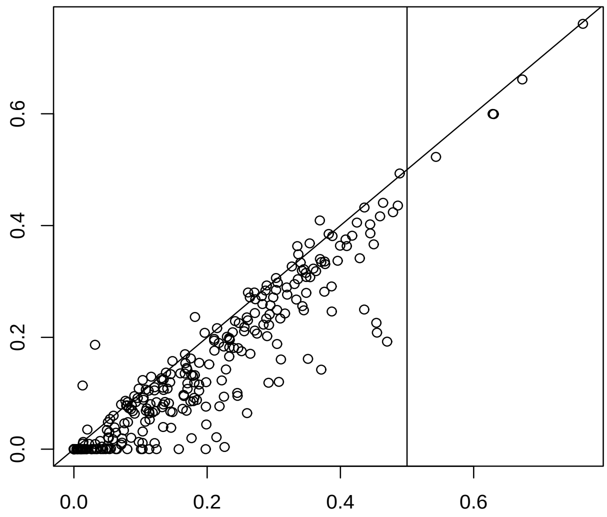

Figure 1: Minimum Centrists, 2000 and 2010, All Metro Areas

"minimum number of centrists" from below. This paper now shows this lower bound to *equal* the minimum number of centrists (decentrists).[1] Somewhat loosely, here we "close the gap" between the two concepts. There is no gap. A similar result obtains for computing the "minimum number of decentrists". We will show that the minimum number of decentrists equals (minus) the "least cumulative ring difference". And so we are also able to bound true (unobservable) centrists not just from below, but also from above.

These bounds are meaningful, and for motivation we briefly illustrate our concepts's appeal to understanding centrism in the US. Note first that a majority of centrists takes decisions that favor the center. These decisions, in turn, reinforce the share of centrists. One implication of centrism is the existence of multiple equilibria. Cities with a majoritarian share of centrists should become more centrist over time; while the opposite is true for cities with a majoritarian share of decentrists. Fig. (1) shows the share of minimum centrists in both 2000 (horizontal axis) and 2010 (vertical axis), for every US metro area. Only four metro areas have a minimum centrists share beyond one half. Consistent with theory, these metro areas do not see their minimum centrist shares contract. Conversely, and also consistent with our theory, metro areas with minimum centrist share short of 1/2 often see it drop further (and with only few exceptions see it rise).

The paper has six sections. Section 2 states the primal problem, of identifying the least number of centrists that fits the given distribution of population across city rings. Section 3 offers trial solutions for two examples that point to the building blocks of the solution for the general city. Section 4 states, and proves, the solution for minimum centrists. Section 5 applies similar reasoning to solving for minimum decentrists. And section 6 concludes.

[1] Jacobs (1961) and Breheny (2007) also use the term "decentrists", though with a very different meaning. For Jacobs, decentrists are those early 20th century urban and regional planners such as Lewis Mumford, Clarence Stein, Henry Wright and Catherine Bauer, who advocated "thinning out large cities" by dispersing their "enterprises and populations into smaller, separated cities or, better yet, towns" (p. 19).

2 Landlord-Tenant Matching

Monocentric City. A closed and monocentric city (as pioneered by Wheaton (1973), Pines/Sadka (1986) and Brueckner (1987)) juts $\widetilde{r}$ units of distance out from the CBD (with $\widetilde{r}$ determined shortly). Commuting costs for a resident living at distance r from the CBD are tr. Ricardian rent q follows $q(r) = t(\widetilde{r} - r)$. The city's overall population is s, and the urban wage is w. Residents consume one unit of housing. Housing is built by profit maximizing investors. One unit of capital k combined with one unit of land yields $h(k)$ units of housing, where $h' > 0$ and $h'' < 0$ (again, Brueckner (1987)).

Housing. If p is the price of capital, investors choose k so as to satisfy the $q(r)h_k(k) = p$ necessary for maximum profit. The optimal capital depends on rent q and price p, and so can be written as $k(t(\widetilde{r} - r), p)$. Let $h(r)$ be shorthand for the building height obtained for this optimal capital choice. Then the city boundary $\widetilde{r}$ is determined by the condition that the housing market clear,

$$\int_0^{\widetilde{r}} a(r)h(r)\,dr \;=\; s, \tag{1}$$

where $a(r)$ is land available in a ring of unit width r units of distance away from the CBD. Ratio $a(r)h(r)/s$, also written $f(r)$, indicates the share of the population commuting from within that ring to the CBD. Correspondingly, $F(r)$ denotes the share of households commuting r or less.[2] Now divide the city into $i = 1, \ldots, n$ concentric rings of equal width (n even) around the CBD, with n large enough to justify treating rent, building height, commuting times etc. as identical across ring i's plots. Housing in ring i is app. $f(r_i)s$. We set $f(r_i)s = b_i$, to conform with the LP notation introduced shortly.[3]

Ownership. Traditional urban modeling has residents own urban housing jointly or treats landlords as absentee. Yet we want to avoid both the traditional "common ownership" or "absentee landlord" setup, lest we assume away the important centrist/decentrist-contest that is at the heart of this paper. We replace either assumption by dividing urban residents in two resident classes, resident landlords and tenants. Each landlord owns one unit of housing (an "apartment") that he resides in himself as well as another apartment that he rents out. These two apartments, to be sure, do not need to locate in the same ring.[4]

Realistically, information on any given landlord's two individual properties must be treated as private. And so we cannot say whether this landlord is a centrist or a decentrist. But (unknown) match matrix $X = (x_{ij})$ collects the frequencies with which the various possible matches between landlords and tenants occur, with row i (column j) indicating the landlord's (tenant's) location. Centrists (decentrists) are those landlords whose average property is closer to (further away from) the center than half the distance from the CBD to the city boundary, $\widetilde{r}/2$. Hence centrists are those for whom

$$(r_i + r_j)/2 \;<\; \widetilde{r}/2 \tag{2}$$

[2] We assume a is continuous in r. As h is (differentiable and hence) continuous in r, so is f.

[3] We will also refer to $f(r_i)$ or $f(r_i)s$ as the city's *shape*, following terminology introduced in Arnott/Stiglitz (1981).

[4] Surely there are many other, often more complex, ways to introduce (i) resident landlords with their (ii) tenants into the city.

or, equivalently, $i + j - 1 < n$.[5] An analogous condition applies to decentrists.[6]

Matching. The previous inequality suggests that centrists (decentrists) are to be associated with entries of X that are located strictly above (below) its counter diagonal, i.e. the diagonal that stretches from X's bottom left corner to its top right one. Moreover, being a centrist (or decentrist) does not depend on which apartment is the owner-occupied one, i or j. We may conveniently suggest that landlords always occupy the ring that is closer to the city center. And so with $i \leq j$, X becomes upper triangular. Now, to capture the overall number of households inhabiting ring i we need to sum over all of X's entries in *both*, row i and column i. The resulting sum must equal ring i's available stock of apartments, b_i. And so ring i's housing constraint reads $\sum_{j=1}^{n}(x_{ij} + x_{ji}) = b_i$.

Linear Program. Summing over all centrist-related entries in X gives $\sum_{i=1}^{n-1}\sum_{j=1}^{n-i} x_{ij}$, the true, yet unknown, number of centrists, l^c. Contrast this with the smallest number of centrists conceivable, $\underline{l}^c$. That latter number bounds the true number of centrists l^c from below. To identify $\underline{l}^c$, we minimize the number of centrists given ring housing constraints and the non-negativity requirements $x_{ij} \geqslant 0$. This translates into the following linear program

$$\min_{x_{ij}} \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} x_{ij} \quad \text{s.t.} \quad \begin{aligned} \sum_{j=1}^{n}(x_{ij} + x_{ji}) &= b_i \quad (i = 1, \ldots, n) \\ x_{ij} &\geqslant 0 \quad (i, j = 1, \ldots, n), \end{aligned} \tag{3}$$

analysis of which is the focus of the next two sections.

3 The Minimum Share of Centrists, in Two Specific Cities

We run two eight-ring city examples on how to solve the linear program (3) next. These are examples to offer some intuition on how a feasible, and even optimal, solution to linear program (3) plays out. But in fact they are much more than just examples. They motivate a trial solution that later will generalize to any given city.

Example City 1. Our first city has "city shape" b = (38, 36, 30, 10, 12, 8, 4, 2). To this city, matrix X_1 in (4), in highlighting eight non-zero entries, suggests one basic feasible solution.[7] We briefly illustrate feasibility. Adding up all entries in row 1 and column 1, for instance, gives $20 + 18 = 38$ or b_1, while adding up all entries in row 7 (consisting of zeros only) and column 7 gives just $0 + 4$ or b_7. Our feasible solution here displays one feature that we might expect of an optimal solution, notably that (4) assigns the maximum possible weight to entries on the counterdiagonal (in red on screen). This forces centrists' numbers down as best as we can. We get $x_{18} = \min\{b_1, b_8\} = 2$. Similarly, $x_{27} = 4$, $x_{36} = 8$ and $x_{45} = 10$.

Put differently, whenever possible we allocate a peripheral apartment in some given outer ring j, $5 \leqslant j \leqslant 8$, to a proprietor who owns her other, second apartment in corresponding

[5]This follows from assuming that residents in ring i commute distance $(i - 0.5)\widetilde{r}/n$.

[6]Note that even as decentrists have properties closer to the city extremes, "extremists" probably is not a better term.

[7]Here, as well as in all other match matrices below, entries with no explicit number attached equal zero.

inner ring $9-j$. This must be a necessary property of a centrist-minimizing allocation. (Suppose that X_1 violated this property, i.e. suppose $x_{18} = 1 < 2 = \min\{38, 2\}$. Since there are no apartments, anywhere, capable of successfully turning a landlord in ring 1 – someone who would otherwise be a centrist – into a decentrist, an opportunity to reduce centrists would have been irrevocably wasted.) At the same time, of course, not all apartments in a given peripheral ring j may be assignable to a landlord in corresponding ring $n-j+1$. In ring $j = 5$, for example, only 10 out of 12 apartments are.

There are $(b_1 - b_8) = 36$ apartments in ring 1 still waiting to be allocated, as are $(b_2 - b_7) = 32$ apartments in ring 2 and $(b_3 - b_6) = 22$ apartments in ring 3. We apportion these remainders to landlords owning both their properties within the same ring. Since any match on the main diagonal accounts for two apartments, we set $x_{11} = (b_1 - b_8)/2 = 18$, $x_{22} = (b_2 - b_7)/2 = 16$ and $x_{33} = (b_3 - b_6)/2 = 11$ (all blue on screen). Note that $x_{44} = 0$, given that $x_{45} = 10$ already and that row 4 and column 4 must add up to $b_4 = 10$. It remains to balance housing in ring 5, by setting x_{55} to 1 (brown on screen). – Now, invoking the simplex algorithm would reveal that the solution set out in (4) above not just is feasible but also: optimal.[8] Instead of going through these details here, we offer a systematic treatment below (in the following section).

$$X_1 = \begin{pmatrix} \mathbf{18} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} \\ & \mathbf{16} & 0 & 0 & 0 & 0 & \mathbf{4} & 0 \\ & & \mathbf{11} & 0 & 0 & \mathbf{8} & 0 & 0 \\ & & & 0 & \mathbf{10} & 0 & 0 & 0 \\ & & & & \mathbf{1} & 0 & 0 & 0 \\ & & 0 & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix} \tag{4}$$

We conclude that the trial number of centrists suggested by (4) also is the minimum number of centrists given the specific city shape b in hand. Adding up these centrists is simple enough. We merely need to collect the few non-zero entries found above the counterdiagonal. These are conveniently located on the upper half of the main diagonal (blue on screen). This gives $\sum_{i=1}^{3}(b_1 - b_{9-i})/2$ or 45 minimum centrists. Minimum centrists' share in city population becomes 45/140. Computing minimum centrists provides valuable information here. It is not possible for the true number of centrists to fall short of 45. But it is quite possible – if not utterly likely – for the true number of centrists to surpass 45. Of course, the latter likely occurs should true matches deviate from one of the optimal solutions.

Example City 2. Our second example city exhibits housing stocks described by "city shape" $b = (38, 14, 30, 10, 12, 8, 26, 2)$. We take an important step towards generalization by introducing the concept of ring difference δ_i now, where $\delta_i = b_i - b_{n+1-i}$ is the number of apartments in "leading" ring i minus that in "lagging" or "antagonist" ring $n+1-i$. It is defined for $1 \leqslant i \leqslant 4$. In our second example city, δ_i is positive for i equal to 1 or 3 (since there we have a "surplus") and it is negative if i equals 2 or 4 (because then there is

[8] It is not, however, a unique optimal solution. For example, letting any landlord trade apartments with her or his tenant would generate another optimal solution.

a “deficit”). Contrast this with our first example city, where all first three ring differences are positive.

True to our strategy of emphasizing the counterdiagonal, feasible solution X_2 in (5) assigns as many apartments as possible in lagging rings to owners in corresponding leading rings. And because we have a surplus in rings 1 and 3, for these rings this works just fine. All apartments in rings 8 and 6 can be assigned to landlords living in rings 1 and 3, respectively. And while this works less well for apartments in lagging rings 5 and 7, remaining apartments are not always lost on us. Ring 2’s deficit (of $-(b_2 - b_7) = 12$), for instance, we may “save up for”, or “post to”, the next best successive ring boasting a surplus. In our example, this is ring 3 (where $b_3 - b_6 = 22$). The 12 apartments reflecting ring 2’s deficit can valuably be employed to offset the better part of ring 3’s surplus.

And so we set entry x_{37} in X_2 to $b_7 - b_2$, or 12 (green). Intuitively, the 12 ring 7-apartments not assignable to ring 2-landlords now are assigned to landlords in ring 3, to at least turn those off centrism. Note that the same is not possible to do with the ring deficit arising in ring 4. There simply are no later rings. – Everything else parallels our discussion of the first example. We balance the first three rings’ housing constraints by setting $x_{11} = (b_1 - b_8)/2 = 18$, $x_{22} = 0$ and $x_{33} = (b_3 - (b_6 + (b_7 - b_2)))/2 = 5$. Again, moreover, the basic feasible solution, set out in (5), also is the optimal one. Minimum centrists are found to sum to 23, if only to see their share in the overall total attain a mere 23/140.

$$X_2 = \begin{pmatrix} \color{blue}{\mathbf{18}} & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{\mathbf{2}} \\ & 0 & 0 & 0 & 0 & 0 & \color{red}{\mathbf{14}} & 0 \\ & & \color{blue}{\mathbf{5}} & 0 & 0 & \color{red}{\mathbf{8}} & \color{green}{\mathbf{12}} & 0 \\ & & & 0 & \color{red}{\mathbf{10}} & 0 & 0 & 0 \\ & & & & \color{brown}{\mathbf{1}} & 0 & 0 & 0 \\ & & 0 & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix} \tag{5}$$

Review. What can be learned from these two examples? We have seen that in both cities minimum centrists may be written as the cumulative sum of the first three ring differences, $\sum_{i=1}^{3}(b_1 - b_{9-i})/2$. This is true even as δ_2 is positive in the first example city while negative in the second. But why does it make sense to include δ_2 in either example? The answer is this: On the one hand, including $\delta_2/2$ in the cumulative sum *when positive* acknowledges the fact that $(b_2 - b_7)/2$ *landlords in ring 2 cannot* be turned away from centrism. On the other hand, including $\delta_2/2$ in the cumulative sum *when negative* acknowledges the fact that $(b_7 - b_2)/2$ *landlords in ring 3 can* (be turned off centrism).

We must also wonder about why $\sum_{i=1}^{3}(b_1 - b_{9-i})/2$ excludes $\delta_4/2$. In particular, why is negative $\delta_4/2$ not included in the second city’s cumulative sum when negative $\delta_2/2$ is? Following our previous intuition, there is no need to “save” ring 5 apartments for later because there are no later surpluses to “swipe away”. The only remaining ring that could possibly feature a centrist landlord is ring 4. Yet here δ_4’s negative sign indicates that the planner can already afford each landlord in ring 4 a ring 5-apartment that successfully counters that landlord's initial impulse to "go centrist". And with no further centrists to collect in the fourth ring, our cumulative sum should: stop short of it.

Tentative Ideas. Two ideas emerge from this: (i) Minimum centrists can be represented as a cumulative sum of successive ring differences. (ii) Successive ring differences should enter that cumulative sum if they are positive. And they should even enter the cumulative sum if they are negative, as long as they can help "wipe out" subsequent positive ones. Negative ring differences should be included if and only if they are followed by positive ones at least equal in size. I.e., the cumulative sum should include successive ring differences as long as this helps raise the cumulative sum. Equivalently, to *minimize* centrists we must *maximize* the cumulative sum of ring differences. We will return to this equivalence in a moment, when generalizing our examples (in the next section).

4 The Minimum Share of Centrists, Anywhere

Primal vs. Dual Program. We allow for any $n \times 1$ vector of ring housing stocks $b = (b_1, \ldots, b_n)$ now, except for ruling out any b_i to equal zero. We then put the corresponding linear program (3) into standard form. We first stack all n columns of X into one long $(n^2 \times 1)$ vector x. This gives $x' = (x_{11}, \ldots, x_{1n}, \ \ldots \ , x_{n1}, \ldots, x_{nn})$. To address the objective function in (3) in matrix notation, let c_i equal an $n \times 1$ vector consisting of ones only except for the last i entries, which are zero instead. For example, c_3 is a list of $n-3$ ones followed by three zeros, i.e. $c_3' = (1, \ldots, 1, 0, 0, 0)$. Define an $n^2 \times 1$ list of weights c by setting $c' = (c_1', \ldots, c_n')$. Then our objective function $\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} x_{ij}$ can be cast as the product $c'x$.

Next, let τ_i denote an $n \times 1$ vector featuring 2 in its i-th row and 1 in all other rows. For example, $\tau_2' = (1, 2, 1, \ldots, 1)$. Moreover, let J_i denote what becomes of the $n \times n$ identity matrix once row i has been replaced with τ_i'. Then the coefficient matrix A is $A = (J_1, \ \ldots \ , J_n)$; it is of dimensions $n \times n^2$. The tableau in Table (1) illustrates A in its bottom left part. This table also indicates our specific vector of objective function weights c (in its first row) as well as the vector of ring housing stocks b (last column).[9]

1	1	1	1	…	1	0	…	1	0	0	0	…	0	0	
2	1	1	1	…	1	1	…	1	0	0	0	…	0	0	b_1
0	1	0	0	…	0	0	…	0	1	0	0	…	0	0	b_2
0	0	1	0	…	0	0	…	0	0	1	0	…	0	0	b_3
				$\vdots$			…					$\vdots$			$\vdots$
0	0	0	0	…	1	0	…	0	0	0	0	…	1	0	b_{n-1}
0	0	0	0	…	0	1	…	1	1	1	1	…	1	2	b_n

Table 1: Matrix A, objective function weights c and housing stocks b

With this extra notation in hand, linear program (3) may equivalently be stated as $\min_x c'x$ subject to $Ax = b$ and $x \geqslant 0$, where the equality constraints may also be read off Table

[9] As inspection of A makes clear, ours is not a transportation problem (e.g., as defined in Hadley (1963)).

(1)'s rows. This program's dual is $\max_y y'b$ such that $y'A \leqslant c'$, where y is the dual's $(n \times 1)$ vector of unknowns, $y' = (y_1, \ \ldots \ , y_n)$. Table (1) also indicates the dual's constraints; these can be read off its columns. For instance, the constraint complementary to x_{11} being strictly positive simply is $2y_1 \leqslant c_{11} = 1$ (see first column in Table (1)).

Rather than immediately analyze the general case, we focus on a seemingly special case first. This case allows us to best connect with the principles that emerge from our discussion of the two example cities (section 3). To address this special case, let us introduce the partial cumulative sum $\Delta(i) = \sum_{j=1}^{i} \delta_j/2$. This sum cumulates successive ring differences δ_j up to ring i, where of course $i \leqslant n/2$. And let index i^* be the index that maximizes this cumulative sum, i.e.

$$i^* \ = \ \arg \ \max_i \ \sum_{j=1}^{i} (b_j \ - \ b_{n+1-j})/2. \tag{6}$$

Our point of departure on the way to the fully general solution is a city for which (i) $\Delta(i^*) > 0$ and (ii) all ring differences δ_i are negative except when $i = i^*$, when $\delta_{i^*} > 0$.

Trial Solution. We set out a basic feasible solution to the primal problem next. Table (2) shows X in tabular form and may be a useful reference as we go along. Again, entries of X never addressed are zero. Moreover, also note the formal resemblance between Table (2) on the one hand and matrices X_1 and X_2 on the other. Now, we begin by considering the elements on the counterdiagonal of match matrix X. Here we set (red on screen)

$$x_{i,n+1-i} \ = \ \min\{b_i, b_{n+1-i}\} \qquad\qquad (i = 1, \ldots, n/2). \tag{7}$$

Given our sign assumptions regarding the δ_i, this entails setting all entries $x_{1,n}$ "up" to $x_{i^*-1,n+2-i^*}$, and again from $x_{i^*+1,n-i^*}$ to $x_{n/2,n/2+1}$, equal to the leading ring's stock, b_i. Only $x_{i^*,n+1-i^*}$ becomes the lagging ring's stock, b_{n+1-i^*}. Note how this assignment makes as many owners of property in leading rings (voters who otherwise likely are centrists) as possible disavow centrism.

Moreover, set (green on screen)

$$x_{i^*,n+1-i} \ = \ (b_{n+1-i} \ - \ b_i) \qquad\qquad (i = 1, \ldots, i^* - 1). \tag{8}$$

Note that the expressions on the r.h.s. represent ring deficits. Deficits originating in rings prior to i^* are posted to leading ring i^*, as the earliest next ring offering up an excess. "Apartment savings" originating in rings up to i^* then are matched up with apartments in ring i^*. This generalizes how we proceeded earlier when setting x_{37} equal to 12 in example city 2.

Next, let (blue on screen)

$$x_{i^*i^*} \ = \ \Big(b_{i^*} \ - \ \big(b_{n+1-i^*} \ + \ \sum_{k=1}^{i^*-1} (b_{n+1-k} \ - \ b_k)\big)\Big)\Big/2, \tag{9}$$

or $\Delta(i^*)$. At first sight nothing seems to preclude $x_{i^*i^*}$ from being strictly negative, in contradiction to primal variables' non-negativity constraints. However, recall that i^*

Ro./Co.	1			i^*				$n/2+1$		$n-i^*$				n
1	0													$x_{1,n}$
		$\ddots$											$\iddots$	0
			0									$x_{i^*-1,n+2-i^*}$	0	0
i^*				x_{i^*,i^*}							$x_{i^*,n+1-i^*}$	$x_{i^*,n+2-i^*}$	$\cdots$	$x_{i^*,n}$
					0					$x_{i^*+1,n-i^*}$				
						$\ddots$			$\iddots$					
$n/2$							0	$x_{n/2,n/2+1}$						
$n/2+1$								$x_{n/2+1,n/2+1}$						
									$\ddots$					
$n-i^*$										$x_{n-i^*,n-i^*}$				
											0			
												0		
													$\ddots$	
n														0

Table 2: Non-Zero Elements in Basic Feasible Solution

maximizes the cumulative sum of ring differences. And so $\sum_{j=1}^{i^*} \delta_j/2 \geqslant 0$, i.e. a non-negative number. And note that this latter number just coincides with the r.h.s. of (9). Put yet differently, ring excess δ_{i^*} is more than sufficient to offset the ring deficits δ_k associated with, and inherited from, all the rings prior to i^*. And so $x_{i^*i^*}$ really is non-negative.

At last we set (brown on screen)

$$x_{n+1-i,n+1-i} \;=\; (b_{n+1-i} \;-\; b_i)/2 \qquad (i = i^* + 1, \ldots, n/2). \tag{10}$$

Ring deficits originating in rings following i^* are relegated to main diagonal elements below the counterdiagonal, to the desirable effect of contributing nothing to the number of centrists. Note how equations (7), (8), (9) and (10) set out a feasible solution of the primal.

Complementary Slackness. We invoke complementary slackness between the primal and the dual. For $i = 1, \ldots, n/2$, entries on the counterdiagonal $x_{i,n+1-i}$ are strictly positive (see (7)), as is the main diagonal element $x_{i^*i^*}$ (see (9)). By complementary slackness, the corresponding constraints of the dual – read off the corresponding columns of Table (1) – must be met with equality, and so

$$y_i \;=\; -\,y_{n+1-i} \qquad (i = 1, \ldots, n/2) \qquad \text{and} \qquad y_{i^*} \;=\; 1/2. \tag{11}$$

These equations specify the weights on ring housing stocks b_i in the dual's objective.

For $i = 1, \ldots, i^* - 1$, entries $x_{i^*,n+1-i}$ are strictly positive, too (see (8)). Again, by complementary slackness, corresponding constraint inequalities in the dual become binding. And so, according to Table (1), $y_{i^*} = -y_{n+1-i}$. Combining this with $y_{n+1-i} = -y_i$ and the fact that $y_{i^*} = 1/2$ (see (11)) gives the first set of equations in (12). At last we make use of equations (10). For $i = i^* + 1, \ldots, n/2$, constraint (in)equalities translate into $y_i = 0$. Joint with the first set of equations in (11), this in turn implies the second set of equations in (12):

$$y_i \ = \ 1/2 \quad (i = 1, \ldots, i^* - 1) \quad \text{and} \quad y_i \ = \ 0 \ \ (i = i^* + 1, \ldots, n - i^*). \tag{12}$$

Table (3) collects the full solution to equations (11) and (12), denoted $\bar{y}$ and easily shown to be feasible, too.

i	1	...	i^*	i^*+1	...	$n-i^*$	$n-i^*+1$	...	n
$\bar{y}_i$	1/2	1/2	1/2	0	0	0	−1/2	−1/2	−1/2

Table 3: The dual's optimal solution

Basic Feasible Solution is Optimal. Let us now put together feasibility and complementary slackness, using standard reasoning in linear programming (Bertsimas/Tsitsiklis (1997)). First, feasibility of $\bar{x}$ and $\bar{y}$ implies $b = A\bar{x}$ and $\bar{y}'A \leqslant c'$, respectively, and hence $\bar{y}'b = \bar{y}'(A\bar{x}) = (\bar{y}'A)\bar{x} \leqslant c'\bar{x}$. Second, complementary slackness implies $(\bar{y}'A - c')\bar{x} = 0$ or $(\bar{y}'A)\bar{x} = c'\bar{x}$. And so we may conclude that $\bar{y}'b = c'\bar{x}$. This in turn implies that $c'\bar{x}$ equals minimum centrists, and hence that $\bar{x}$ solves (3). Of course, if $\bar{x}$ is optimal, then so is $\bar{y}$, justifying Table (3)'s title.

We compute the objective function values for primal and dual, providing a check on optimality of $\bar{x}$ and $\bar{y}$ as well as, of course, the desired minimum number of centrists itself. On the one hand, summing over all entries above the counter diagonal the objective function value in the primal gives $x_{i^*i^*}$ as on the r.h.s. of equation (9). But then:

$$\boxed{\underline{l}^c \ = \ \Delta(i^*) \ = \ \max_i \sum_{j=1}^{i} (b_j \ - \ b_{n+1-j})/2.} \tag{13}$$

On the other hand, computing the sum of ring stocks using the optimal weights in (11) and (12) yields the very same formula, i.e. $\sum_{j=1}^{i^*}(b_j - b_{n+1-j})/2$. This formula represents the optimal value of both primal and dual. Thus it also represents the minimum conceivable number of centrists. We briefly pause to appreciate the formula's generality: the greatest cumulative ring difference gives a universal closed form solution for minimum centrists. It provides an observer of an arbitrary given city with a prediction of centrists' minimum.

Our proof is for a city whose ring differences, with the exception of δ_{i^*}, are all negative (also see the first two rows in Table (4) in the Appendix). The Appendix shows how the proof quickly generalizes. Subsections 8.2 through 8.4 show that our results in essence remain unchanged as some, or even all, ring differences exhibit an arbitrary sign. Formula (13) remains valid throughout. This is quite straightforward since also accounting for positive ring differences (Appendix) is simpler than accounting for negative ones (this section): witness solution X_1 as opposed to X_2 (in section 3). Now, translating minimum centrist numbers in formula (13) into minimum centrists' share in all landlords, by dividing $\Delta(i^*)$ by $s/2$, gives the following variant of this result:

Proposition 1: (Greatest Cumulative Ring Difference and Centrists)

Centrists' minimum conceivable share of the landlord population, $\underline{\lambda}^c$, is given by the greatest cumulative ring difference, $\underline{\lambda}^c = \max_i \sum_{j=1}^{i} \left(b_j/s - b_{n+1-j}/s\right)$.

5 Centrists vs. Decentrists

Minimum Decentrists. We bring in decentrists now. Intuitively, where before we have used b_{n+1-i} to "swipe away" or "swamp" potential centrists in i (as best as we could), conversely we now use b_i to "swamp" decentrists in $n+1-i$ (as best as we can). Applying a proof similar to that in section 4 (omitted for brevity), we find that minimum decentrists correspond to: minus the least cumulative ring difference. That is, if $i^{**} = \arg\max_i \sum_{j=1}^{i}(-(b_j - b_{n+1-j}))/2$, then minimum decentrists $\underline{l}^d$ are equal to

$$\boxed{\underline{l}^d \;=\; -\Delta(i^{**}) \;=\; -\min_i \sum_{j=1}^{i} (b_j \;-\; b_{n+1-j})/2}\,. \tag{14}$$

Translating this number into a share gives

Proposition 2: (Least Cumulative Ring Difference and Decentrists)

Decentrists' minimum conceivable share of the landlord population, $\underline{\lambda}^d$, is given by minus the least cumulative ring difference, $\underline{\lambda}^d = -\min_i \sum_{j=1}^{i} \left(b_j/s - b_{n+1-j}/s\right)$.

Upper Bounds. We quickly turn lower bounds in Propositions 1 and 2 into corresponding upper bounds. Subtracting centrists from overall landlord population $s/2$ gives the sum of decentrists and indifferent landlords. This in turn is the sum of all elements of X strictly below or on the counter diagonal. The following linear program looks for the maximum sum of decentrists/indifferents:

$$\max_{x_{ij}} \left(s/2 - \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} x_{ij}\right) \quad \text{s.t.} \quad \begin{aligned} \sum_{j=1}^{n}(x_{ij} + x_{ji}) &= b_i \quad (i = 1,\ldots,n) \\ x_{ij} &\geqslant 0 \quad (i,j = 1,\ldots,n). \end{aligned} \tag{15}$$

Comparing linear programs, clearly the maximizer to (15) coincides with the minimizer to (3). But this implies that $s/2 - \underline{l}^c$ is the maximum conceivable number of decentrists/indifferents. And so $s/2 - \underline{l}^c$ is an upper bound to decentrists only (Proposition, Part (ii)). A similar argument suggests that $s/2 - \underline{l}^d$, where $\underline{l}^d$ is the minimum number of decentrists, is an upper bound to centrists (Proposition 3, Part (i)).

Proposition 3: (Upper Bounds on Centrists and Decentrists)

(i) λ^c is bounded from above by $1 - \underline{\lambda}^d$. (ii) λ^d is bounded from above by $1 - \underline{\lambda}^c$.

6 Conclusions

The paper provides (novel) estimators of centrists and decentrists. These estimators are highly general, and are simple to compute from easily observable data for any city. We suggest that conflicts between centrists and decentrists could help explain other contested policies beyond decentralization. Examples include carbon taxation, rationing central city land, urban growth boundaries, decentralizing retail, tightening building height limits, implementing minimum lot size, or introducing road tolls. To these policy fields, the paper's greatest lower bounds on centrists and decentrists are applicable, too.

7 Literature

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8 Appendix

8.1 Cumulative Ring Difference

We introduce some extra notation first. Consider the cumulative sum $\Delta(h)$, for some h between 1 and $n/2$. Suppose $\Delta(h)$ is preceded by some other cumulative sum $\Delta(g)$ that is greater than it, i.e. $\Delta(h) < \Delta(g)$ for $g < h$.[10] Borrowing terminology established in the context of the "Rising Sun Lemma" (Spivak (1994)), then we will say that the cumulative sum up to, and including, ring difference h is "in the shadow of" the cumulative sum up to, and including, ring difference g.[11] Of course, there will be rings that are never overshadowed. Among those, ring i^*, defined in equation (6), is the one exhibiting the greatest cumulative sum, $\Delta(i^*)$. For the two numerical cities in section 3, to give an example, $i^* = 3$.

8.2 Not All Ring Differences Negative

In the main text we took the first step towards a fully general analysis. Our point of departure was the city of the type spelt out in Table (4). The table header has the ring difference index i, the second row provides ring difference δ_i's sign, and the third row indicates whether or not the corresponding cumulative ring difference $\Delta(i)$ is overshadowed ($\bullet$ is a suggestive shorthand) or not ($\circ$). As mentioned above, in this city all ring differences both prior to i^* and beyond $i^* + 1$ are negative and overshadowed.

i	1	2	3	$\dots$	$i^* - 1$	i^*	$i^* + 1$	$i^* + 2$	$\dots$	$n/2$
δ_i	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$
$\Delta(i)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

Table 4: A Parametric City

Nothing of substance changes if one (or more) of those shadow differences is (are) positive, rather than negative. To see this we turn to the city set out in Table (5) below, with the second ring the one ring to have flipped its sign. We assume that everything else remains the same, and so $\Delta(2) < \Delta(0)$ while i^* keeps maximizing $\Delta(i)$.[12]

i	1	2	3	$\dots$	$i^* - 1$	i^*	$i^* + 1$	$i^* + 2$	$\dots$	$n/2$
δ_i	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$
$\Delta(i)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet$

Table 5: Negative and Positive Ring Differences

[10] We define $\Delta(0) \equiv 0$. Even the first ring may be overshadowed, by $\Delta(0)$, if $\delta_1 < 0$.

[11] In our first example city, the cumulative ring difference at 4 is overshadowed (by the cumulative ring difference at 3, say), while in the second example city cumulative ring differences at 2 and 4 are (by cumulative ring differences 1 and 3, respectively, for example).

[12] These assumptions are not restrictive. First, if $\Delta(0) < \Delta(2)$, we would have to consider alternating spells of ring differences in the shadow and not in the shadow. This case is considered shortly. And second, if i^* shifted due to δ_2 flipping its sign, nothing would change in the argument below as long as $2 < i^*$.

We introduce the following three, i.e. not numerous, changes to the primal's solution: (i) Entry $x_{2,n-1}$ ceases to be b_2 and turns into b_{n-1} instead. (ii) Entry x_{2n} becomes $(b_2 - b_{n-1})$, replacing the zero it was before. (iii) Entry $x_{i^*,n}$ drops from ring difference $(b_n - b_1)$ to the "difference of ring differences" $(b_n - b_1) - (b_2 - b_{n-1})$. These changes maintain feasibility, as is easily checked by consulting the housing constraints of the four rings affected.

Note that $x_{i^*i^*}$ is not among the entries changed. This particular entry continues to equal $\Delta(i^*)/2$. Since this entry is the only one to enter the primal objective's optimal value, our formula does not change either. Note the role of ring 2 still being overshadowed here. While δ_2 is positive, it is not sufficiently so to offset the negative δ_1 that precedes it. And hence $(b_n - b_1) - (b_2 - b_{n-1})$ or $x_{i^*,n}$ indeed is strictly positive. Now let us check the implied changes for the dual. Since $x_{2,n-1}$ and $x_{i^*,n}$ continue to exceed zero, complementary constraints of the dual continue to be binding. And since x_{2n} now also exceeds zero, the corresponding dual constraint becomes binding, so that $y_2 = -y_n$. This we knew before, and so this extra equation is redundant. We conclude that formula $\Delta(i^*)/2$ continues to apply. Of course, the objective's numerical value changes.

Exploring a sign change for any other ring difference, or for additional ring differences, proceeds along similar lines. That is, formula $\Delta(i^*)/2$ continues to capture the minimum number of centrists whatever the signs of the ring differences in rings up to i^*, as long as these ring differences are overshadowed.

8.3 Not All Ring Differences Overshadowed

What (if anything) changes if one (or more) of the ring differences are not overshadowed? Let us allow for the possibility that not all ring differences prior to i^* are overshadowed, as in Table (6). Let all ring differences from 1 up to $i' - 1$ be in the shadow of ring 0, and all ring differences between $i' + 1$ and $i^* - 1$ be overshadowed by i', so that i^* is not in the shadow. One optimal feasible solution assigns $\sum_{j=1}^{i'} \delta_j/2$ to $x_{i'i'}$, and $\sum_{j=i'+1}^{i^*} \delta_j/2$ to $x_{i^*i^*}$, and zero to any other element above the counterdiagonal. The corresponding minimum number of centrists becomes the sum of these two (only non-zero) terms. But this is just our familiar $\Delta(i^*)/2$. Adding extra spells of ring differences in the shadow adds nothing of substance here.

i	1	...	$i'-1$	i'	$i'+1$	...	i^*-1	i^*	i^*+1	...	$n/2$
$\Delta(i)$	•	•	•	○	•	•	•	○	•	•	•

Table 6: Alternating Spells of Shadow and Light

At last we turn to the question of what happens if any ring differences following $i^* + 1$ (rather than preceding i^*) exhibit a positive sign. Recall that, by definition of i^*, ring differences beyond i^* must be overshadowed. Let one of these ring differences be positive, rather than negative, i.e. $i^* + 2$ say. Being in the shadow of i^*, the excess arising in ring difference $i^* + 2$ is "swamped" by the deficit in the previous ring difference at $i^* + 1$. Once more, there is no change in the number of minimum centrists.