On the Growth Effects of North-South Trade: The Role of Labor Market Flexibility Technical Appendix on Transitional Dynamics

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This technical appendix contains the proof of Theorem 2 in our paper "On the Growth Effects of North-South Trade: The Role of Labor Market Flexibility". It is demonstrated that for initial values $\xi(0)$ close to ξ^* and $L^N(0)$ close to L^{N*} there exists a unique trajectory converging to the steady state (g^*, ξ^*, L^{N*}) and that this convergent path may feature either monotonic convergence or damped oscillations:

Theorem 2. *The system of linear differential equations obtained by linearizing the system (6), (7), (8) in a neighborhood of its steady state possesses a unique convergent path. Depending on parameter values, this convergent path is monotonic or oscillatory.*

The proof is difficult. We proceed in several steps. Let $\tilde{y}(t) \equiv y(t) - y^*$ for $y \in \{g, \xi, L^N\}$. Then:

Result 1. *The linearized version of the system (6), (7), (8) is:*

$$\begin{pmatrix} \dot{\tilde{g}} \\ \dot{\tilde{\xi}} \\ \dot{\tilde{L}}^N \end{pmatrix} = \begin{pmatrix} J^*_{\dot{g}g} & J^*_{\dot{g}\xi} & J^*_{\dot{g}L^N} \\ \frac{m}{g^*+m} & -(g^*+m) & 0 \\ ma & 0 & -(\beta+m) \end{pmatrix} \begin{pmatrix} \tilde{g} \\ \tilde{\xi} \\ \tilde{L}^N \end{pmatrix}, \tag{T.1}$$

where

$$J^*_{\dot{g}g} = m + (g^* + m + \rho)\left(1 + \frac{\alpha}{1-\alpha}\frac{g^*}{g^*+m}\right)$$

$$J^*_{\dot{g}\xi} = \frac{\alpha}{1-\alpha}(g^* + m + \rho)^2$$

$$J^*_{\dot{g}L^N} = -\frac{(\beta + m) + (g^* + m + \rho)}{a}.$$

Proof: Let $J_{\dot{y}z} \equiv \partial\dot{y}/\partial z$ for $y, z \in \{g, \xi, L^N\}$. Then

$$J^*_{\dot{g}g} = m - \underbrace{\left[\rho + m + g^* - \frac{1-\alpha}{\alpha\xi^*}\left(\frac{L^{N*}}{a} - g^*\right)\right]}_{=0} + \left(\frac{L^{N*}}{a} - g^*\right)\left(1 + \frac{1-\alpha}{\alpha\xi^*}\right)$$

$$
\begin{aligned}
&= m + \underbrace{\frac{1-\alpha}{\alpha\xi^*}\left(\frac{L^{N*}}{a} - g^*\right)}_{=\rho+m+g^*} + \underbrace{\left(\frac{L^{N*}}{a} - g^*\right)}_{=(\rho+m+g^*)\frac{\alpha\xi^*}{1-\alpha}} \\
&= m + (g^* + m + \rho)\left(1 + \frac{\alpha}{1-\alpha}\xi^*\right) \\
&= m + (g^* + m + \rho)\left(1 + \frac{\alpha}{1-\alpha}\frac{g^*}{g^* + m}\right) \\
J^*_{\dot{g}\xi} &= \left(\frac{L^{N*}}{a} - g^*\right)\frac{1-\alpha}{\alpha\xi^{*2}}\left(\frac{L^{N*}}{a} - g^*\right) \\
&= \frac{\alpha}{1-\alpha}\underbrace{\left[\frac{1-\alpha}{\alpha\xi^*}\left(\frac{L^{N*}}{a} - g^*\right)\right]^2}_{=(\rho+m+g^*)^2} \\
&= \frac{\alpha}{1-\alpha}(g^* + m + \rho)^2 \\
J^*_{\dot{g}L^N} &= -\frac{\beta+m}{a} + \frac{1}{a}\underbrace{\left[\rho + m + g^* - \frac{1-\alpha}{\alpha\xi^*}\left(\frac{L^{N*}}{a} - g^*\right)\right]^2}_{=0} - \frac{1}{a}\underbrace{\frac{1-\alpha}{\alpha\xi^*}\left(\frac{L^{N*}}{a} - g^*\right)}_{=\rho+m+g^*} \\
&= -\frac{\beta+m}{a} - \frac{g^* + m + \rho}{a} \\
&= -\frac{(\beta+m) + (g^* + m + \rho)}{a} \\
J^*_{\dot{\xi}g} &= 1 - \xi^* = \frac{m}{g^* + m} \\
J^*_{\dot{\xi}\xi} &= -(g^* + m) \\
J^*_{\dot{\xi}L^N} &= 0 \\
J^*_{\dot{L}^N g} &= ma \\
J^*_{\dot{L}^N \xi} &= 0 \\
J^*_{\dot{L}^N L^N} &= -(\beta + m).
\end{aligned}
$$

Q.E.D.

The system contains two state variables ($\tilde{\xi}$ and $\tilde{L}^N$) and one jump variable ($\tilde{g}$). So in order for a unique convergent path to exist, exactly two eigenvalues of the system must have negative real parts. To prove this, we let $\varphi \equiv (\tilde{g}, \tilde{\xi}, \tilde{L}^N)'$ and denote the Jacobian matrix in (T.1) as J^*. Then (T.1) can be rewritten as $\dot{\varphi} = J^*\varphi$. Suppose there exist solutions to this system of the form $\varphi(t) = be^{qt}$, where $b = (b_g, b_\xi, b_{L^N})'$. Then $\dot{\varphi} = qbe^{qt} = q\varphi$. Hence $J^*\varphi = q\varphi$ or, letting I be the identity matrix, $(J^* - qI)\varphi = 0$. Non-trivial solutions $\varphi \neq 0$ exist if and only if $|J^* - qI| = 0$, that is

$$
0 = -q^3 + Tr\, J^* q^2 - B\, J^* q + Det\, J^* \equiv f(q),
$$

where:

Result 2.

$$
Tr\, J^* = \frac{\alpha}{1-\alpha}\frac{g^*(g^* + m + \rho)}{g^* + m} + \rho - \beta
$$

is the trace of the Jacobian,

$$Det\,J^* = (g^*+m+\rho)\left\{\beta\left(\frac{g^*}{1-\alpha}+m\right)+\frac{\alpha}{1-\alpha}m\left[g^*+\frac{(\beta+m)(g^*+m+\rho)}{g^*+m}\right]\right\}.$$

is the determinant and

$$\begin{aligned} B\,J^* &= -m(g^*+m)-g^*(g^*+m+\rho)-\rho(\beta+m)\\ &\quad -(g^*+2m+\beta)(g^*+m+\rho)\frac{\alpha}{1-\alpha}\frac{g^*}{g^*+m}\\ &\quad -\frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)^2}{g^*+m}. \end{aligned}$$

Proof:

$$\begin{aligned} Tr\,J^* &= J^*_{\dot g g}-(g^*+m)-(\beta+m)\\ &= m+\frac{\alpha}{1-\alpha}\frac{g^*(\rho+m+g^*)}{g^*+m}+\rho+m+g^*-(g^*+m)-(\beta+m)\\ &= \frac{\alpha}{1-\alpha}\frac{g^*(g^*+m+\rho)}{g^*+m}+\rho-\beta. \end{aligned}$$

$$\begin{aligned} Det\,J^* &= J^*_{\dot g g}(g^*+m)(\beta+m)\\ &\quad +J^*_{\dot g L^N}ma(g^*+m)\\ &\quad +J^*_{\dot g\xi}\frac{m}{g^*+m}(\beta+m)\\ &= (\beta+m)\left[m(g^*+m)+(g^*+m+\rho)\left(\frac{g^*}{1-\alpha}+m\right)\right]\\ &\quad -[(\beta+m)+(g^*+m+\rho)]m(g^*+m)\\ &\quad +\frac{\alpha}{1-\alpha}(g^*+m+\rho)^2\frac{m}{g^*+m}(\beta+m)\\ &= (\beta+m)\left[m(g^*+m)+(g^*+m+\rho)\left(\frac{g^*}{1-\alpha}+m\right)\right.\\ &\quad \left.+\frac{\alpha}{1-\alpha}(g^*+m+\rho)^2\frac{m}{g^*+m}-m(g^*+m)\right]-m(g^*+m)(g^*+m+\rho)\\ &= (\beta+m)\left[(g^*+m+\rho)\left(\frac{g^*}{1-\alpha}+m\right)+\frac{\alpha}{1-\alpha}(g^*+m+\rho)^2\frac{m}{g^*+m}\right]\\ &\quad -(g^*+m+\rho)m(g^*+m)\\ &= (g^*+m+\rho)\left\{(\beta+m)\left[\left(\frac{g^*}{1-\alpha}+m\right)+\frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)}{g^*+m}\right]-m(g^*+m)\right\}\\ &= (g^*+m+\rho)\left[\beta\left(\frac{g^*}{1-\alpha}+m\right)+m\frac{\alpha}{1-\alpha}g^*+(\beta+m)\frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)}{g^*+m}\right]\\ &= (g^*+m+\rho)\left\{\beta\left(\frac{g^*}{1-\alpha}+m\right)+\frac{\alpha}{1-\alpha}m\left[g^*+\frac{(\beta+m)(g^*+m+\rho)}{g^*+m}\right]\right\}. \end{aligned}$$

$$B\,J^* = \begin{vmatrix} J^*_{\dot g g} & J^*_{\dot g \xi}\\ \frac{m}{g^*+m} & -(g^*+m)\end{vmatrix} + \begin{vmatrix} J^*_{\dot g g} & J^*_{\dot g L^N}\\ ma & -(\beta+m)\end{vmatrix} + \begin{vmatrix} -(g^*+m) & 0\\ 0 & -(\beta+m)\end{vmatrix}$$

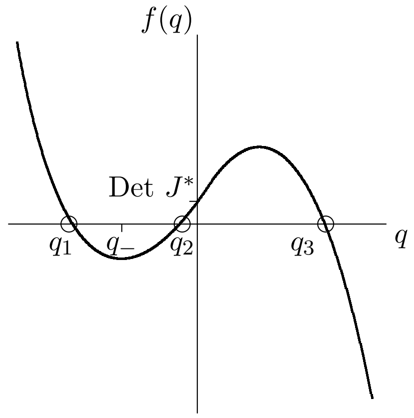

Figure T.1: The characteristic roots

$$
\begin{aligned}
&= -J^*_{\dot{g}g}(g^*+m+\beta+m) - J^*_{\dot{g}\xi}\frac{m}{g^*+m} - J^*_{\dot{g}L^N}ma + (g^*+m)(\beta+m) \\
&= -(g^*+m+\beta+m)\left[m+(g^*+m+\rho)\left(1+\frac{\alpha}{1-\alpha}\frac{g^*}{g^*+m}\right)\right] \\
&\quad -\frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)^2}{g^*+m} + m[(\beta+m)+(g^*+m+\rho)] + (g^*+m)(\beta+m) \\
&= -(g^*+m+\underbrace{\beta+m}_{(1)})m - g^*(g^*+m+\rho) - \underbrace{m(g^*+m+\rho)}_{(2)} - (\beta+m)(\underbrace{g^*}_{(3)}+\underbrace{m}_{(4)}+\rho) \\
&\quad -(g^*+m+\beta+m)(g^*+m+\rho)\frac{\alpha}{1-\alpha}\frac{g^*}{g^*+m} - \frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)^2}{g^*+m} \\
&\quad +\underbrace{m(\beta+m)}_{(1)} + \underbrace{m(g^*+m+\rho)}_{(2)} + \underbrace{g^*(\beta+m)}_{(3)} + \underbrace{m(\beta+m)}_{(4)} \\
&= -m(g^*+m) - g^*(g^*+m+\rho) - \rho(\beta+m) \\
&\quad -(g^*+2m+\beta)(g^*+m+\rho)\frac{\alpha}{1-\alpha}\frac{g^*}{g^*+m} \\
&\quad -\frac{\alpha}{1-\alpha}\frac{m(g^*+m+\rho)^2}{g^*+m}.
\end{aligned}
$$

Q.E.D.

From $f(0) = Det\, J^* > 0$ and $f'(0) = -B\, J^* > 0$, it follows that there is exactly one positive real root, say q_3 (see figure T.1). The other two roots, q_1 and q_2 say, are either real and negative or complex conjugates. Theorem 2 asserts that:

Result 3. *If q_1 and q_2 are complex conjugates, the real part is not positive.*

Proof: Rewrite $f(q)$ as

$$f(q) = (q_1-q)(q_2-q)(q_3-q) = -q^3 + (q_1+q_2+q_3)q^2 - [q_1q_2 + q_3(q_1+q_2)]q + q_1q_2q_3.$$

Suppose q_1 and q_2 are complex conjugates with positive real part: $q_{1/2} = \gamma \mp \delta i$ with $\gamma > 0$. Then the coefficient of q in the characteristic equation is negative:

$$-[q_1q_2 + q_3(q_1+q_2)] = -(\gamma^2 + \delta^2 + q_3 2\gamma) < 0.$$

This contradicts $B\,J^* < 0$ and thus proves that exactly two eigenvalues have negative real parts.[1] Q.E.D.

To each eigenvalue q_j corresponds a particular solution $\varphi(t) = b_j e^{q_j t}$ of (T.1), where $b_j = (b_{jg}, b_{j\xi}, b_{jL^N})'$ is the eigenvector associated with the eigenvalue q_j. From $(J^* - q_j I)\varphi = 0$, it follows that $(J^* - q_j I)b_j = 0$ or, spelled out in detail,

$$\begin{pmatrix} J^*_{\dot{g}g} - q_j & J^*_{\dot{g}\xi} & J^*_{\dot{g}L^N} \\ \frac{m}{g^*+m} & -(g^*+m+q_j) & 0 \\ ma & 0 & -(\beta+m+q_j) \end{pmatrix} \begin{pmatrix} b_{jg} \\ b_{j\xi} \\ b_{jL^N} \end{pmatrix} = 0.$$

Eliminating $b_{j\xi}$ and b_{jL^N} yields

$$b_j = b_{jg} \begin{pmatrix} 1 \\ \frac{m}{(g^*+m)(g^*+m+q_j)} \\ \frac{ma}{\beta+m+q_j} \end{pmatrix}. \tag{T.2}$$

The general solution of (T.1) is obtained by combining the particular solutions $\varphi(t) = b_j e^{q_j t}$ for $j = 1, 2, 3$ linearly: $\varphi(t) = \sum_{j=1}^{3}(B_j/b_{jg})b_j e^{q_j t}$, where the B_j's ($j = 1, 2, 3$) are arbitrary constants. Since we are interested in the behavior of the convergent growth path, the coefficient of the particular solution associated with the unstable eigenvalue q_3 has to be set equal to zero: $B_3/b_{3g} = 0$ and $\varphi(t) = \sum_{j=1}^{2}(B_j/b_{jg})b_j e^{q_j t}$. Evaluating this equation at $t = 0$ and inserting (T.2) yields:

$$\begin{pmatrix} \tilde{g}(0) \\ \tilde{\xi}(0) \\ \tilde{L}^N(0) \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ \frac{m}{(g^*+m)(g^*+m+q_1)} \\ \frac{ma}{\beta+m+q_1} \end{pmatrix} + B_2 \begin{pmatrix} 1 \\ \frac{m}{(g^*+m)(g^*+m+q_2)} \\ \frac{ma}{\beta+m+q_2} \end{pmatrix}.$$

Since the initial values $\xi(0)$ and $L^N(0)$ are given, this is a system of algebraic equations in $\tilde{g}(0)$, B_1 and B_2. Solving for $\tilde{g}(0)$ yields the initial growth rate:

Result 4. *The initial growth rate $\tilde{g}(0)$ satisfies*

$$\tilde{g}(0) = \frac{1}{m(\beta - g^*)} \left\{ \left[\prod_{j=1}^{2}(\beta + m + q_j)\right] \frac{\tilde{L}^N(0)}{a} - \left[\prod_{j=1}^{2}(g^* + m + q_j)\right] (g^* + m)\tilde{\xi}(0) \right\}.$$

Proof: First eliminate $B_2 = \tilde{g}(0) - B_1$:

$$\begin{aligned} \tilde{\xi}(0) &= \frac{mB_1}{(g^*+m)(g^*+m+q_1)} - \frac{m[B_1 - \tilde{g}(0)]}{(g^*+m)(g^*+m+q_2)} \\ \tilde{L}^N(0) &= \frac{maB_1}{\beta+m+q_1} + \frac{ma[\tilde{g}(0) - B_1]}{\beta+m+q_2}. \end{aligned}$$

[1] This can also be proved by applying the Routh-Hurwitz Theorem, which states that the number of eigenvalues with positive real parts is equal to the number of sign changes in the scheme $-1 \,||\, Tr\,J^* \,||\, -B\,J^* + Det\,J^*/Tr\,J^* \,||\, Det\,J^*$. If $Tr\,J^* > 0$ the sign scheme is $-||+||+||+$, so there is one sign change and hence one unstable eigenvalue. If $Tr\,J^* < 0$, the sign scheme is $-||-||?||+$. Again there is one sign change and one unstable eigenvalue.

Equivalently:

$$\left[\prod_{j=1}^{2}(g^*+m+q_j)\right]\frac{g^*+m}{m}\tilde{\xi}(0) = (q_2-q_1)B_1+(g^*+m+q_1)\tilde{g}(0)$$

$$\left[\prod_{j=1}^{2}(\beta+m+q_j)\right]\frac{\tilde{L}^N(0)}{ma} = (q_2-q_1)B_1+(\beta+m+q_1)\tilde{g}(0).$$

Subtracting the former equation from the latter yields:

$$(\beta-g^*)\tilde{g}(0)=\left[\prod_{j=1}^{2}(\beta+m+q_j)\right]\frac{\tilde{L}^N(0)}{ma}-\left[\prod_{j=1}^{2}(g^*+m+q_j)\right]\frac{g^*+m}{m}\tilde{\xi}(0).$$

Division by $\beta-g^*$ gives the formula in Result 4. Q.E.D.

If the stable eigenvalues q_1 and q_2 are real, then the elements of b_1 and b_2 as well as B_1 and B_2 are real. So $\varphi(t)=\sum_{j=1}^{2}(B_j/b_{jg})b_je^{q_jt}$ implies that $\tilde{g}$, $\tilde{\xi}$ and $\tilde{L}^N$ converge monotonically to zero. If, on the other hand, the stable eigenvalues are complex conjugates, that is $q_{1/2}=\gamma\mp\delta i$ with $\gamma<0$, then the three variables display damped oscillations. For instance:

Result 5. *The adjustment of the rate of innovation obeys*

$$\tilde{g}(t)=e^{\gamma t}\left[\tilde{g}(0)\cos(\delta t)-\frac{z-\gamma\tilde{g}(0)}{\delta}\sin(\delta t)\right],$$

where $z\equiv[\prod_{j=1}^{2}(g^*+m+q_j)](g^*+m)\tilde{\xi}(0)/g^*-(g^*+m)\tilde{g}(0)$ *is a (real-valued) constant.*

Proof: Given $\tilde{g}(0)$, either one of the two formulas for $\tilde{\xi}(0)$ and $\tilde{L}^N(0)$ can be used to solve for B_1. Taking the first one, one obtains:

$$\begin{aligned} B_1 &= \frac{1}{q_2-q_1}\left\{\left[\prod_{j=1}^{2}(g^*+m+q_j)\right]\frac{g^*+m}{m}\tilde{\xi}(0)-(g^*+m+q_1)\tilde{g}(0)\right\} \\ &= \frac{z-\tilde{g}(0)q_1}{q_2-q_1}, \end{aligned}$$

where

$$z\equiv\left[\prod_{j=1}^{2}(g^*+m+q_j)\right]\frac{g^*+m}{m}\tilde{\xi}(0)-(g^*+m)\tilde{g}(0)$$

is a real-valued constant. Moreover,

$$\begin{aligned} B_2 &= \tilde{g}(0)-B_1 \\ &= \frac{q_1-q_2}{q_1-q_2}\tilde{g}(0)-B_1 \\ &= \frac{z-\tilde{g}(0)q_2}{q_1-q_2}. \end{aligned}$$

The convergent growth path obeys $\varphi(t)=\sum_{j=1}^{2}(B_j/b_{jg})b_je^{q_jt}$, hence $\tilde{g}(t)=B_1e^{q_1t}+B_2e^{q_2t}$. Inserting the formulas for B_1 and B_2 derived above, we have

$$\tilde{g}(t)=\frac{z-\tilde{g}(0)q_1}{q_2-q_1}e^{q_1t}+\frac{z-\tilde{g}(0)q_2}{q_1-q_2}e^{q_2t}.$$

Now suppose the stable eigenvalues q_1 and q_2 are complex conjugates: $q_{1/2} = \gamma \mp \delta i$ (with $\gamma < 0$). Then

$$\begin{aligned}
\tilde{g}(t) &= \frac{z - \tilde{g}(0)q_1}{\underbrace{q_2 - q_1}_{=2\delta i}} e^{q_1 t} + \frac{z - \tilde{g}(0)q_2}{\underbrace{q_1 - q_2}_{=-2\delta i}} e^{q_2 t} \\
&= \frac{z - (\gamma - \delta i)\tilde{g}(0)}{2\delta i} e^{(\gamma - \delta i)t} - \frac{z - (\gamma + \delta i)\tilde{g}(0)}{2\delta i} e^{(\gamma + \delta i)t} \\
2\delta i e^{-\gamma t}\tilde{g}(t) &= [z - (\gamma - \delta i)\tilde{g}(0)] \underbrace{e^{-\delta i t}}_{\substack{= \cos(\delta t) \\ -i\sin(\delta t)}} - [z - (\gamma + \delta i)\tilde{g}(0)] \underbrace{e^{\delta i t}}_{\substack{= \cos(\delta t) \\ +i\sin(\delta t)}} \\
&= \{[z - (\gamma - \delta i)\tilde{g}(0)] - [z - (\gamma + \delta i)\tilde{g}(0)]\} \cos(\delta t) \\
&\quad - \{[z - (\gamma - \delta i)\tilde{g}(0)] + [z - (\gamma + \delta i)\tilde{g}(0)]\} i \sin(\delta t) \\
&= 2\delta i \tilde{g}(0)\cos(\delta t) - 2[z - \gamma\tilde{g}(0)] i \sin(\delta t) \\
\tilde{g}(t) &= e^{\gamma t}\left[\tilde{g}(0)\cos(\delta t) - \frac{z - \gamma\tilde{g}(0)}{\delta}\sin(\delta t)\right].
\end{aligned}$$

Q.E.D.

To prove Theorem 2, it remains to show that the stable eigenvalues q_1 and q_2 may in fact be real (as illustrated in Figure T.1) or complex. This is done by way of example. Let q_- denote the value of q for which $f(q)$ attains its local minimum ($f'(0) = -B\,J^* > 0$ implies that a minimum exists). q_1 and q_2 are real if $f(q_-) < 0$ and complex otherwise.

Result 6. *q_- is given by*

$$q_- = \frac{Tr\,J^*}{3} - \sqrt{\left(\frac{Tr\,J^*}{3}\right)^2 - \frac{B\,J^*}{3}}.$$

Proof: Equating the derivative of $f(q)$ to zero yields $f'(q) = -3q^2 + 2Tr\,J^* q - B\,J^* = 0$ or

$$q^2 - \frac{2Tr\,J^*}{3} q + \frac{B\,J^*}{3} = 0.$$

q_- is the smaller solution of this quadratic equation. Since $B\,J^* < 0$, the smaller solution is given by:

$$q_- = \frac{Tr\,J^*}{3} - \sqrt{\left(\frac{Tr\,J^*}{3}\right)^2 + \frac{B\,J^*}{3}}.$$

Q.E.D.

Result 7. *The negative eigenvalues q_1 and q_2 may be real or complex.*

Proof: Examples for both cases are easily found. Suppose $\alpha = 0.5$, $\beta = 1$, $\rho = 0.02$. Further, assume that $\bar{L}^N/a$ is such that $g^* = 0.03$. Finally, let $m = 0.1$. Then $Tr\,J^* = -0.94538$, $Det\,J^* = 0.04349$ and $B\,J^* = -0.09938$. Hence $q_- = -0.67904$ and $f(q_-) = -0.14680 < 0$. In this example, q_1 and q_2 are real. Now, everything else equal, let $m = 1$. Then $Tr\,J^* = -0.94942$, $Det\,J^* = 3.28528$ and $B\,J^* = -2.26455$, so that $q_- = -1.24114$ and $f(q-) = 0.92403 > 0$. Here, the stable eigenvalues are complex and the dynamics are cyclical. In both examples constructed above, the consistency requirement (10) is satisfied if L^S is sufficiently great. For the sake of completeness, it may be noticed that, from (9), $\bar{L}^N/a = 0.06808$ in the first example and $\bar{L}^N/a = 0.09117$ in the second one. Q.E.D.

This completes the proof of Theorem 2. Q.E.D.